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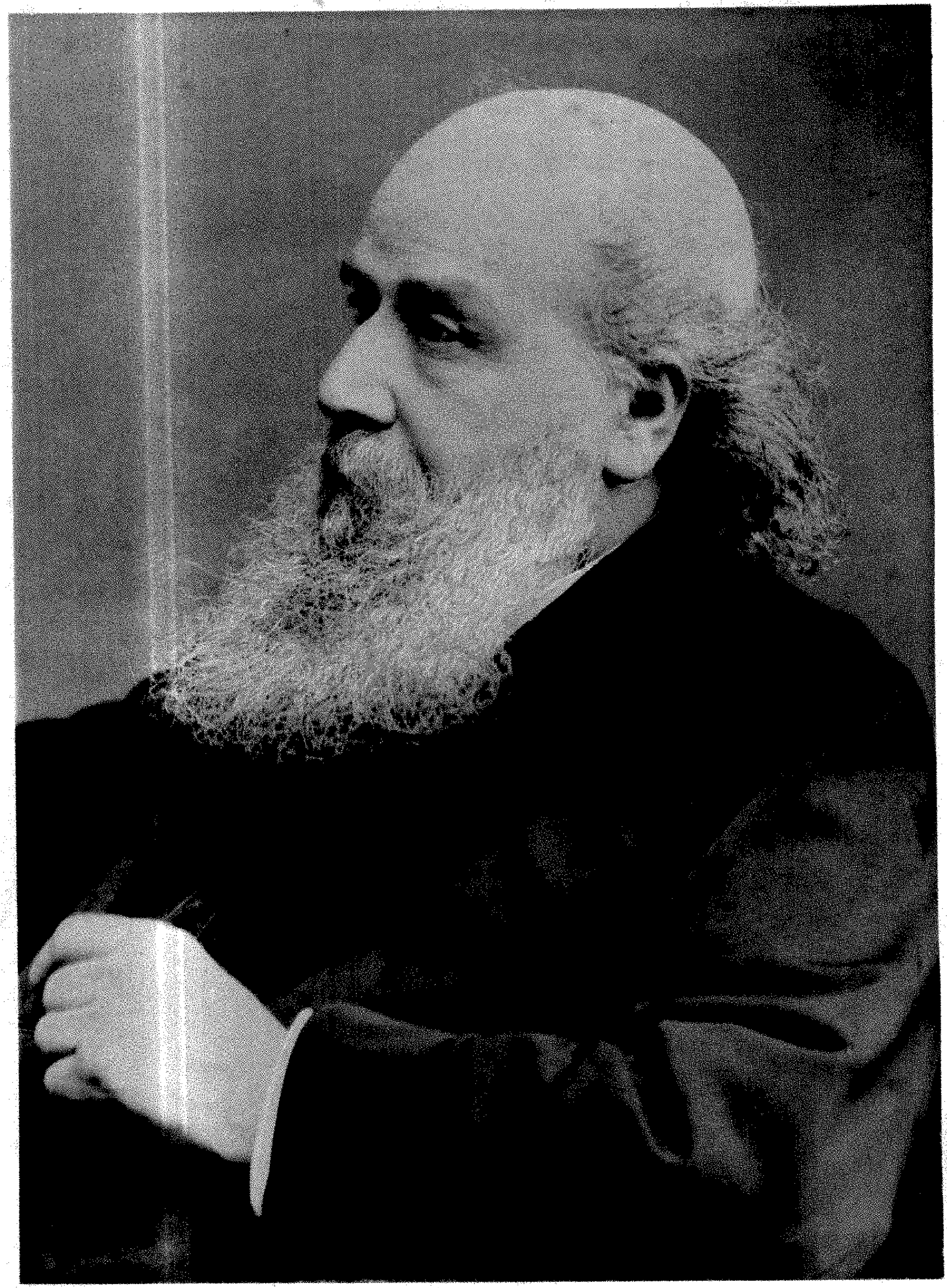

Yours very truly

Lectures on the Theory of Reciprocants.

By Professor Sylvester, F. R. S., *Savilian Professor of Geometry in the University of Oxford.*

[Reported by James Hammond, M. A.]

LECTURE XXXIII.

In this Lecture it is proposed to investigate the differential equation of a cubic curve having a given absolute invariant $\frac{S^3}{T^2}$.

Since the value of $\frac{S^3}{T^2}$ is the same for any homographic transformation of the cubic as for the original curve, the differential equation in question must be of the form

$$\text{Plenarily absolute principiant} = \frac{S^3}{T^2}.$$

This equation is (as we see at once by differentiating it) the integral of another of the form

$$\text{Principiant} = 0,$$

which is satisfied, independently of the value of the absolute invariant, at all points on a perfectly general cubic.

Now, the differential equation of the general cubic is of the 9th order, and when expressed in terms of A, B, C, ... contains no letter beyond E. Hence the integral of this equation, which we are in search of, will be of the 8th order and will contain no capital letter beyond D.

When no letters beyond D are involved, all plenarily absolute principiants are functions of the two fundamental, or protomorphic, ones,

$$\frac{AC-B^2}{A^{\frac{8}{3}}}, \quad \frac{A^2D-3ABC+2B^3}{A^4}.$$

Thus the differential equation of a cubic with a given absolute invariant is of the form

$$F\left(\frac{AC-B^2}{A^{\frac{8}{3}}}, \frac{A^2D-3ABC+2B^3}{A^4}\right) = \frac{S^3}{T^2}.$$

M. Halphen actually integrates the differential equation of the general cubic, which he shows (on p. 52 of his Thèse sur les Invariants Différentiels) may be put under the form

$$\xi\zeta d\xi + \left\{\zeta - \frac{3}{8}(\xi + 3)(\xi + 27)\right\} d\zeta = 0,$$

where, in our notation,

$$\xi = \frac{24\,(A^2D - 3ABC + 2B^3)}{A^4},\ \zeta = \frac{288\,(AC - B^2)^3}{A^8}.$$

The integral of this equation, which M. Halphen obtains partly from geometrical considerations, involves an arbitrary parameter depending on $\frac{S^3}{T^2}$. His result is as follows:

$$R^2 = hQ^3,$$

where

$$2^9R = 2^9\zeta^3 + 2^6.3\,[(\xi - 3^2)^2 + 2^4.3^4]\,\zeta^2 + 2^3.3\,(\xi + 3^3)^3(\xi - 3^2.5)\,\zeta + (\xi + 3^3)^6,$$

$$2^6Q = 2^6\zeta^2 + 2^4(\xi + 3^3)(\xi - 3^2.5)\,\zeta + (\xi + 3^3)^4,$$

and

$$T^2 - 64hS^3 = 0.$$

(Two misprints, which are here corrected, occur in the expression for R as given on p. 54 of the Thèse.)

In this result the invariant S differs in sign from the invariant usually denoted by that letter. Thus the discriminant is $T^2 - 64S^3$ instead of $T^2 + 64S^3$.

When $h = 1$ the discriminant vanishes and the differential equation becomes

$$R^2 - Q^3 = 0.$$

This is divisible by a numerical multiple of ζ^3; in fact,

$$R^2 = Q^3 + 2^3.3^5\zeta^3P,$$

where

$$2^6P \equiv (2^3\zeta + \xi^2 - 2.3^3\xi - 3^5)^2 + 2^6.3\xi^3 = 0$$

is the differential equation of a nodal cubic, previously obtained by Halphen.

It is from a knowledge of the fact that $P = 0$ and another algebraic relation between ξ and ζ, which he finds by trial to be $Q = 0$, constitute two particular integrals of the differential equation to the general cubic, that he arrives, not by any regular method but by repeated strokes of penetrative genius, at the general integral

$$R^2 = hQ^3.$$

In establishing the relation $T^2 - 64hS^3 = 0$ he supposes that, by means of the equation to the cubic and its differentials as far as the 8th order inclusive, the coefficients of the cubic have been expressed in terms of the variables x, y and the derivatives of y with respect to x up to the 8th order, and that the values thus obtained for the coefficients have been substituted in Aronhold's S and T.

The abbreviations introduced by the use of our notation enable us to actually perform this calculation, which would otherwise be impracticable in consequence of the enormous amount of labor required; and we shall use this method to obtain the plenarily absolute principiant which, equated to $\frac{S^3}{T^2}$, gives the differential equation to a cubic with a known absolute invariant.

Using the symbolic notation explained in Lecture XXXII, the equation of the cubic and its first eight differentials are

$$
\begin{aligned}
&u^3 = 0,\\
&u^2u_1 = 0,\\
&2uu_1^2 + u^2u_2 = 0,\\
&2u_1^3 + 6uu_1u_2 + u^2u_3 = 0,\\
&3u_1^2(2.1) + 6u_1u(3.1) + 3u_1(3.2) + 3u^2(4.1) + 3u(4.2) + (4.3) = 0,\\
&3u_1^2(3.1) + 6u_1u(4.1) + 3u_1(4.2) + 3u^2(5.1) + 3u(5.2) + (5.3) = 0,\\
&3u_1^2(4.1) + 6u_1u(5.1) + 3u_1(5.2) + 3u^2(6.1) + 3u(6.2) + (6.3) = 0,\\
&3u_1^2(5.1) + 6u_1u(6.1) + 3u_1(6.2) + 3u^2(7.1) + 3u(7.2) + (7.3) = 0,\\
&3u_1^2(6.1) + 6u_1u(7.1) + 3u_1(7.2) + 3u^2(8.1) + 3u(8.2) + (8.3) = 0,
\end{aligned}
$$

where $\quad u = p + qx + y,\ u_1 = q + t,\ u_2 = 2a,\ u_3 = 6b;$

as usual, $\quad t = \frac{dy}{dx},\ a = \frac{1}{2}\cdot\frac{d^2y}{dx^2},\ b = \frac{1}{6}\cdot\frac{d^3y}{dx^3},\ \ldots\ldots;$

$(m.\mu)$ denotes the coefficient of h^m in $(ah^2 + bh^3 + ch^4 + \ldots)^\mu$; and if, as in Salmon's Higher Plane Curves (2d edit., p. 187), the equation of the cubic is taken to be

$$r + 3a_0x + 3a_1y + 3b_0x^2 + 6b_1xy + 3b_2y^2 + c_0x^3 + 3c_1x^2y + 3c_2xy^2 + c_3y^3 = 0,$$

then, in the above equations, the symbols

$$p^3,\ p^2q,\ p^2,\ pq^2,\ pq,\ p\ ,\ q^3,\ q^2,\ q\ ,\ 1$$

stand for $\quad r,\ a_0\ ,\ a_1,\ b_0\ ,\ b_1\ ,\ b_2\ ,\ c_0,\ c_1\ ,\ c_2,\ c_3.$

These nine equations are sufficient to determine the values of the coefficients of the cubic which have to be substituted in $\frac{S^3}{T^2}$ in order to obtain our differential equation, which will be, as we have seen, of the form

$$F\left(\frac{AC - B^2}{A^{\frac{8}{3}}},\ \frac{A^2D - 3ABC + 2B^3}{A^4}\right) = \frac{S^3}{T^2}.$$

Since this equation *contains nothing which involves* x, y, *or* t, these letters must have disappeared spontaneously in the process of forming it, and consequently we may, at any stage of the work, *give* x, y, *and* t *any arbitrary values without thereby affecting the result.* Let, then,

$$x = 0,\ y = 0,\ t = 0,\ \text{so that}\ u = p,\ u_1 = q,\ u_2 = 2a,\ u_3 = 6b,$$

and the first four equations become

$$\begin{aligned} u^3 &= p^3 = r = 0, \\ u^2u_1 &= p^2q = a_0 = 0, \\ \frac{1}{2}(2uu_1^2 + u^2u_2) &= pq^2 + p^2a = b_0 + a_1a = 0, \\ \frac{1}{2}(2u_1^3 + 6uu_1u_2 + u^2u_3) &= q^3 + 6pqa + 3p^2b = c_0 + 6b_1a + 3a_1b = 0. \end{aligned}$$

Writing in the last five equations

$$\begin{aligned} u_1^2 &= q^2 = c_1, \\ u_1u &= pq = b_1, \\ u_1 &= q = c_2, \\ u^2 &= p^2 = a_1, \\ u &= p = b_2, \\ 1 &= c_3, \end{aligned}$$

we have

$$\begin{aligned} 3c_1(2.1) + 6b_1(3.1) + 3c_2(3.2) + 3a_1(4.1) + 3b_2(4.2) + c_3(4.3) &= 0, \\ 3c_1(3.1) + 6b_1(4.1) + 3c_2(4.2) + 3a_1(5.1) + 3b_2(5.2) + c_3(5.3) &= 0, \\ 3c_1(4.1) + 6b_1(5.1) + 3c_2(5.2) + 3a_1(6.1) + 3b_2(6.2) + c_3(6.3) &= 0, \\ 3c_1(5.1) + 6b_1(6.1) + 3c_2(6.2) + 3a_1(7.1) + 3b_2(7.2) + c_3(7.3) &= 0, \\ 3c_1(6.1) + 6b_1(7.1) + 3c_2(7.2) + 3a_1(8.1) + 3b_2(8.2) + c_3(8.3) &= 0.^* \end{aligned}$$

Substituting in $\frac{S^3}{T^2}$ for r, a_0, b_0, c_0 their values given by the equations

$$r = 0,\ a_0 = 0,\ b_0 + a_1a = 0,\ c_0 + 6b_1a + 3a_1b = 0,$$

and for the mutual ratios of a_1, b_1, b_2, c_1, c_2, c_3 their values found by solving the last five equations, we obtain the differential equation required.

---

[*] These equations are only set out for the sake of distinctness; when our abbreviations are introduced, only two terms survive in the first three, and only three terms in the last two of these five equations.

Referring to Salmon's Higher Plane Curves, p. 188, we see that, when $r = 0$,

$$S = (c^2a^2) + (cb^2a) - (b^2)^2,$$
$$T = 4\,(c^3a^3) - 3\,(c^2b^2a^2) - 12\,(b^2)(cb^2a) + 8\,(b^2)^3,$$

where (c^2a^2), (cb^2a), ... are functions of a_0, a_1, b_0, b_1, b_2, c_0, c_1, c_2, c_3, which, when $a_0 = 0$, become

$$(c^2a^2) = (c_0c_2 - c_1^2)\,a_1^2,$$
$$(cb^2a) = (b_0^2c_3 - 3b_0b_1c_2 + b_0b_2c_1 + 2b_1^2c_1 - b_1b_2c_0)\,a_1,$$
$$(b^2) = b_0b_2 - b_1^2,$$
$$(c^3a^3) = (c_0^2c_3 - 3c_0c_1c_2 + 2c_1^3)\,a_1^3,$$
$$\begin{aligned}(c^2b^2a^2) = (c_0^2b_2^2 &- 4c_0c_1b_1b_2 - 2c_0c_2b_0b_2 - 4c_0c_2b_1^2 + 8c_0c_3b_0b_1 \\ &+ 8c_1^2b_1^2 + 4c_1^2b_0b_2 - 12c_1c_2b_0b_1 - 8c_1c_3b_0^2 + 9c_2^2b_0^2)\,a_1^2.\end{aligned}$$

We have now reached a point at which the work will be greatly facilitated by the introduction of the capital letters A, B, C, D. This is usually done by writing for

$$\begin{matrix} a, & b, & c, & d, & e, & f, & g, \\ 1, & 0, & 0, & A, & B, & C, & D + \frac{25}{8}A^2. \end{matrix}$$

But in the present instance we may make a further simplification by writing

$$A = 1,\ B = 0,\ C = C_1,\ D = D_1,$$

for the only effect of this will be to make the final result take the form

$$F(C_1, D_1) = \frac{S^3}{T^2}$$

instead of
$$F\left(\frac{AC - B^2}{A^{\frac{8}{3}}}, \frac{A^2D - 3ABC + 2B^3}{A^4}\right) = \frac{S^3}{T^2}.$$

The form of the function will not be affected by writing in it $A = 1$, $B = 0$, and the letters A, B can be restored at pleasure by making

$$C_1 = \frac{AC - B^2}{A^{\frac{8}{3}}},\ D_1 = \frac{A^2D - 3ABC + 2B^3}{A^4}.$$

Hence we may write for

$$\begin{matrix} a, & b, & c, & d, & e, & f, & g, \\ 1, & 0, & 0, & 1, & 0, & C_1, & D_1 + \frac{25}{8}. \end{matrix}$$

Instead of the coefficient of

$$h^m \text{ in } (ah^2 + bh^3 + ch^4 + \ldots.)^\mu,$$

$(m.\mu)$ will now signify

$$\text{co. } h^m \text{ in} \left\{ h^2 + h^5 + C_1 h^7 + \left(D_1 + \frac{25}{8} \right) h^8 \right\}^\mu .$$

Thus we have

$$\begin{array}{lll}
(2.1) = 1 & & \\
(3.1) = 0 & (3.2) = 0 & \\
(4.1) = 0 & (4.2) = 1 & (4.3) = 0, \\
(5.1) = 1 & (5.2) = 0 & (5.3) = 0, \\
(6.1) = 0 & (6.2) = 0 & (6.3) = 1, \\
(7.1) = C_1 & (7.2) = 2 & (7.3) = 0, \\
(8.1) = D_1 + \frac{25}{8} & (8.2) = 0 & (8.3) = 0.
\end{array}$$

Hence the equations which give $a_1, b_1, b_2, c_1, c_2, c_3$ become

$$\begin{gathered}
c_1 + b_2 = 0, \\
c_2 + a_1 = 0, \\
6b_1 + c_3 = 0, \\
c_1 + a_1 C_1 + 2b_2 = 0, \\
2b_1 C_1 + 2c_2 + a_1 \left(D_1 + \frac{25}{8} \right) = 0.
\end{gathered}$$

From the first four of these, coupled with the equations

$$b_0 + a_1 = 0, \quad c_0 + 6b_1 = 0,$$

obtained by making $a = 1$ and $b = 0$ in the original equations which give b_0, c_0, we find

$$\begin{aligned}
c_0 = \quad c_3 &= -6b_1, \\
c_1 = -b_2 &= -C_1^2, \\
c_2 = \quad b_0 &= -a_1 = C_1,
\end{aligned}$$

by assuming $a_1 = -C_1$ (which we are at liberty to do since any one of the coefficients may be chosen arbitrarily).

The last equation then gives

$$b_1 = \frac{D_1}{2} + \frac{9}{16}.$$

Substituting these values in the previously given expressions for (c^2a^2), (cb^2a), ... we have

$$\begin{aligned}
(c^2a^2) &= -(6b_1 + C_1^3) C_1^3, \\
(cb^2a) &= -(4b_1^2 - 9b_1 - C_1^3) C_1^3, \\
(b^2) &= C_1^3 - b_1^2, \\
(c^3a^3) &= (216b_1^3 + 18b_1 C_1^3 + 2C_1^6) C_1^3, \\
(c^2b^2a^2) &= (312b_1^3 + 20b_1^2 C_1^3 - 24b_1 C_1^3 + 9C_1^3 + 4C_1^6) C_1^3.
\end{aligned}$$

Hence $S = (c^2a^2) + (cb^2a) - (b^2)^2$

$$= -C_1^6 + 3b_1C_1^3 - 2b_1^2C_1^3 - b_1^4,$$

and $T = 4(c^3a^3) - 3(c^2b^2a^2) - 12(b^2)(cb^2a) + 8(b^2)^3$

$$= -8C_1^9 - 3(8b_1^2 - 12b_1 + 9)C_1^6 - 12b_1^3(2b_1 - 3)C_1^3 - 8b_1^6.$$

To express S and T in terms of A, B, C, D, we write

$$C_1 = \frac{AC - B^2}{A^{\frac{8}{3}}},\ b_1 = \frac{D_1}{2} + \frac{9}{16} = \frac{A^2D - 3ABC + 2B^3}{2A^4} + \frac{9}{16},$$

or, if we use Halphen's notation in which

$$\zeta = \frac{288(AC - B^2)^3}{A^8},\ \xi = \frac{24(A^2D - 3ABC + 2B^3)}{A^4},$$

we have $$2^5.3^2C_1^3 = \zeta,\ 2^4.3b_1 = \xi + 3^3,$$

and consequently,

$$2^3.3\ (2b_1 - 3) = \xi - 3^2.5;$$

$$2^5.3^2(8b_1^2 - 12b_1 + 9) = (2^4.3b_1 - 2^2.3^2)^2 + 2^4.3^4 = (\xi - 3^2)^2 + 2^4.3^4.$$

Hence

$$-2^{16}.3^4S = 2^{16}.3^4C_1^6 + 2^{16}.3^4b_1(2b_1 - 3)C_1^3 + 2^{16}.3^4b_1^4$$

$$= 2^6\zeta^2 + 2^4(\xi + 3^3)(\xi - 3^2.5)\zeta + (\xi + 3^3)^4,$$

$$-2^{21}.3^6T = 2^{24}.3^6C_1^9 + 2^{21}.3^7(8b_1^2 - 12b_1 + 9)C_1^6 + 2^{23}.3^7b_1^3(2b_1 - 3)C_1^3 + 2^{24}.3^6b_1^6$$

$$= 2^9\zeta^3 + 2^6.3\left[(\xi - 3^2)^2 + 2^4.3^4\right]\zeta^2 + 2^3.3(\xi + 3^3)^3(\xi - 3^2.5)\zeta + (\xi + 3^3)^6,$$

where the expressions on the right-hand side are 2^6Q and 2^9R in Halphen's notation. Thus $$-2^{10}.3^4S = Q,\ -2^{12}.3^6T = R;$$

so that $$\frac{Q^3}{R^2} = -\frac{2^{30}.3^{12}S^3}{2^{24}.3^{12}T^2} = -\frac{64S^3}{T^2}.$$

This result agrees exactly with Halphen's, if we remember that his S is taken with a different sign from ours.

Since $$b_1 = \frac{D_1}{2} + \frac{9}{16} = \frac{A^2D - 3ABC + 2B^3}{2A^4} + \frac{3^2}{2^4},$$

we may write

$$\Phi = 2^4A^4b_1 = 2^3(A^2D - 3ABC + 2B^3) + 3^2A^4,$$

and in like manner

$$\Psi = A^8C_1^3 = (AC - B^2)^3.$$

Now $$2^8A^8(b_1^2 + C_1^3) = \Phi^2 + 2^8\Psi,$$

which is divisible by A^2. Hence if

$$\Phi^2 + 2^8\Psi = A^2\Theta,$$

we have
$$\begin{aligned}\Theta &= 2^8A^6(b_1^2+C_1^3)\\ &= 2^6(A^2D^2-6ABCD+4AC^3+4B^3D-3B^2C^2)\\ &\quad+2^4.3^2A^2(A^2D-3ABC+2B^3)+3^4A^6.\end{aligned}$$

The equations which give S and T in terms of b_1 and C_1 may be written
$$-S=(b_1^2+C_1^3)^2-3b_1C_1^3,$$
$$-T=2^3(b_1^2+C_1^3)^3-2^2.3^2(b_1^2+C_1^3)b_1C_1^3+3^3C_1^6,$$
and consequently,
$$-2^{16}A^{12}S=\Theta^2-2^{12}.3\Phi\Psi,$$
$$-2^{21}A^{18}T=\Theta^3-2^{11}.3^2\Theta\Phi\Psi+2^{21}.3^3A^2\Psi^2,$$
where Θ, Φ, Ψ are the rational integral principiants
$$\begin{aligned}\Theta &= 2^6(A^2D^2-6ABCD+4AC^3+4B^3D-3B^2C^2)\\ &\qquad\qquad+2^4.3^2A^2(A^2D-3ABC+2B^3)+3^4A^6,\\ \Phi &= 2^3(A^2D-3ABC+2B^3)+3^2A^4,\\ \Psi &= (AC-B^2)^3,\end{aligned}$$
which, as we have seen, are connected by the relation
$$\Phi^2+2^8\Psi=A^2\Theta.$$

The differential equation of cubics with a given absolute invariant is
$$\frac{(\Theta^2-2^{12}.3\Phi\Psi)^3}{(\Theta^3-2^{11}.3^2\Theta\Phi\Psi+2^{21}.3^3A^2\Psi^2)^2}=-\frac{2^6S^3}{T^2},$$
or, as it may also be written,
$$(\Theta^2-2^{12}.3\Phi\Psi)^3T^2+2^6S^3(\Theta^3-2^{11}.3^2\Theta\Phi\Psi+2^{21}.3^3A^2\Psi^2)^2=0.$$
For a nodal cubic, the discriminant $T^2+2^6S^3$ vanishes. Hence the differential equation of a nodal cubic is
$$(\Theta^3-2^{11}.3^2\Theta\Phi\Psi+2^{21}.3^3A^2\Psi^2)^2-(\Theta^2-2^{12}.3\Phi\Psi)^3=0.$$
When expanded, and divided by $2^{22}.3^3\Psi^2$, this reduces to
$$A^2\Theta^3-\Theta^2\Phi^2-2^{11}.3^2A^2\Theta\Phi\Psi+2^{14}\Phi^3\Psi+2^{20}.3^3A^4\Psi^2=0,$$
which (since $A^2\Theta-\Phi^2=2^8\Psi$) divides out by $2^8\Psi$, giving
$$\Theta^2-2^3.3^2A^2\Theta\Phi+2^6\Phi^3+2^{12}.3^3A^4\Psi=0,$$
or, what is the same thing,
$$\Theta^2-2^3.3^2A^2\Theta\Phi+2^6\Phi^3+2^4.3^3A^4(A^2\Theta-\Phi^2)=0.$$
This may also be written in the form
$$(\Theta-2^2.3^2A^2\Phi+2^3.3^3A^6)^2+2^6(\Phi-3^2A^4)^3=0,$$
or, replacing Θ and Φ by their values in terms of A, B, C, D,
$$\begin{aligned}&\{2^6(A^2D^2-6ABCD+4AC^3+4B^3D+3B^2C^2)-2^4.3^2A^2(A^2D-3ABC+2B^3)-3^3A^6\}^2\\ &\qquad\qquad+2^{15}(A^2D-3ABC+2B^3)^3=0.\end{aligned}$$

For a cubic whose invariant S vanishes, the differential equation is

$$\Theta^2 - 2^{12}.3\Phi\Psi = 0,$$

and for a cubic whose invariant T vanishes,

$$\Theta^3 - 2^{11}.3^2\Theta\Phi\Psi + 2^{21}.3^3A^2\Psi^2 = 0.$$

For the cuspidal cubic, both S and T vanish, so that the algebraic equation of the cuspidal cubic is a particular solution of each of these equations. We can, however, replace the system

$$\Theta^2 - 2^{12}.3\Phi\Psi = 0, \qquad (1)$$

$$\Theta^3 - 2^{11}.3^2\Theta\Phi\Psi + 2^{21}.3^3A^2\Psi^2 = 0, \qquad (2)$$

by another pair of equations, for one of which the cuspidal cubic is a particular solution, and for the other the complete primitive.

Multiplying the first equation by Θ and subtracting the second from it, we have, after dividing by $2^{11}.3\Psi$,

$$\Theta\Phi - 2^{10}.3^2A^2\Psi = 0. \qquad (3)$$

From (1) and (3) we obtain

$$\Theta^2\Phi^2 = 2^{12}.3\Phi^3\Psi = 2^{20}.3^4A^4\Psi^2.$$

Hence $$\Phi^3 = 2^8.3^3A^4\Psi. \qquad (4)$$

But $$A^2\Theta = \Phi^2 + 2^8\Psi,$$

so that $$A^2\Theta\Phi = \Phi^3 + 2^8\Phi\Psi.$$

Substituting in this the values of $\Theta\Phi$ and Φ^3 found from (3) and (4) and dividing by Ψ, we have $$2^{10}.3^2A^4 = 2^8.3^3A^4 + 2^8\Phi,$$

which gives $$\Phi = 3^2A^4. \qquad (5)$$

Substituting this value of Φ in (4) and rejecting the factor 3^3A^4, we obtain

$$3^3A^8 = 2^8\Psi;$$

i. e. $$\left(\frac{A}{2}\right)^8 = \left(\frac{AC - B^2}{3}\right)^3.$$

In the course of the work we have only rejected powers of Ψ (*i. e.* of $AC - B^2$) and of A, of which neither corresponds to the cuspidal cubic.

Since $\Phi = 3^2A^4$, it follows that $A^2D - 3ABC + 2B^3 = 0$. The equation to the cuspidal cubic above obtained is a particular solution of this, its complete primitive being (see Lecture XXXI) $\gamma = X^\lambda Z^{1-\lambda}$, where λ is an arbitrary constant.

LECTURE XXXIV.

The preceding 33 lectures contain the substance of the lectures on Reciprocants actually delivered, entire or in abstract, in the course of three terms, to a class at the University of Oxford.

A good deal of material remains over which the lecturer has lacked leisure or energy to throw into form, which he hopes to be able to recover and annex to what has gone before as supplemental matter in the convenient form of lectures numbered on from those which have already appeared.

The one that follows is entirely due to Mr. Hammond, who has rendered invaluable aid in compiling, and in many cases bettering, the lectures previously published.

It constitutes probably the most difficult problem in elimination which has been effected up to the present time. J. J. S.

The problem in question is to obtain the differential equation corresponding to the complete primitive

$$(l'x + m'y + n') = (lx + my + n)^{\lambda}(l''x + m''y + n'')^{1-\lambda}$$

(say $Y = X^{\lambda}Z^{1-\lambda}$) by the process of eliminating all the arbitrary constants except λ.

The eliminations to be performed become greatly simplified by aid of the following Lemma. If X be any linear function of x and y, and M_a the absolute pure reciprocant corresponding to M; then

$$X_3 - 4M_aX_1 = 0,$$

where $$\frac{dX}{dx} = a^{\frac{1}{3}}X_1, \quad \frac{dX_1}{dx} = a^{\frac{1}{3}}X_2, \quad \frac{dX_2}{dx} = a^{\frac{1}{3}}X_3.$$

For if we suppose $$X = lx + my + n,$$

two successive differentiations give

$$a^{\frac{1}{3}}X_1 = l + mt$$

and $$a^{\frac{2}{3}}X_2 + a^{-\frac{2}{3}}bX_1 = 2ma.$$

Writing the second of these equations in the form

$$a^{-\frac{1}{3}}X_2 + a^{-\frac{5}{3}}bX_1 = 2m,$$

and differentiating again, we find

$$X_3 - a^{-\frac{4}{3}}bX_2 + a^{-\frac{4}{3}}bX_2 + (4ac - 5b^2)\,a^{-\frac{8}{3}}X_1 = 0,$$

or, since $4M_a = (4ac - 5b^2)\,a^{-\frac{8}{3}}$,

$$X_3 + 4M_aX_1 = 0.$$

N. B.—Throughout the following work all letters with numerical suffixes are to be considered as derived from the corresponding unsuffixed letters in the same way as, in what precedes, X_1, X_2, and X_3 are derived from X; viz. by successive differentiations, each of which is accompanied by a division by $a^{\frac{1}{3}}$.

Writing the equation $\qquad Y = X^\lambda Z^{1-\lambda}$

(in which X, Y, Z denote any three linear functions of x, y) in the form

$$\log Y = \lambda \log X + (1-\lambda) \log Z,$$

we obtain by differentiation and division by $a^{\frac{1}{3}}$,

$$\frac{Y_1}{Y} = \lambda \frac{X_1}{X} + (1-\lambda) \frac{Z_1}{Z}. \tag{1}$$

Let now

$$X_1 = uX,$$
$$Y_1 = vY,$$
$$Z_1 = wZ,$$

so that (1) takes the form

$$v = \lambda u + (1-\lambda) w,$$

and consequently

$$v_1 = \lambda u_1 + (1-\lambda) w_1,$$
$$v_2 = \lambda u_2 + (1-\lambda) w_2.$$

By means of the Lemma it can be shown that

$$u^3 + 3uu_1 + u_2 + 4M_a u = 0, \tag{2}$$
$$v^3 + 3vv_1 + v_2 + 4M_a v = 0, \tag{3}$$
$$w^3 + 3ww_1 + w_2 + 4M_a w = 0. \tag{4}$$

For, since $\qquad X_1 = Xu,$

we have $\qquad X_2 = X_1 u + X u_1 = X(u^2 + u_1)$

and $\qquad X_3 = X_2 u + 2X_1 u_1 + X u_2 = X(u^3 + 3uu_1 + u_2).$

Substituting these values for X_1 and X_3 in

$$X_3 + 4M_a X_1 = 0,$$

we obtain $\qquad u^3 + 3uu_1 + u_2 + 4M_a u = 0,$

which proves equation (2). The equations (3) and (4) connecting v, v_1, v_2 and w, w_1, w_2 are similarly established. We now write

$$\left.\begin{aligned} u + v + w &= 3\omega \\ u - w &= 3z \end{aligned}\right\}$$

These, combined with $\qquad v = \lambda u + (1-\lambda) w,$

give
$$\left.\begin{aligned} u &= \omega - (\lambda - 2)z \\ v &= \omega - (1 - 2\lambda)z \\ w &= \omega - (\lambda + 1)z \end{aligned}\right\}$$

which, when operated on by $a^{-\frac{1}{3}}\frac{d}{dx}$ twice in succession, yield

$$\left.\begin{aligned} u_1 &= \omega_1 - (\lambda - 2)z_1 \\ v_1 &= \omega_1 - (1 - 2\lambda)z_1 \\ w_1 &= \omega_1 - (\lambda + 1)z_1 \end{aligned}\right\} \qquad \left.\begin{aligned} u_2 &= \omega_2 - (\lambda - 2)z_2 \\ v_2 &= \omega_2 - (1 - 2\lambda)z_2 \\ w_2 &= \omega_2 - (\lambda + 1)z_2 \end{aligned}\right\}$$

When expressed in terms of ω, ω_1, ω_2 and z, z_1, z_2, equations (2), (3), and (4) become transformed into

$$P - (\lambda - 2)Q + (\lambda - 2)^2 R - (\lambda - 2)^3 z^3 = 0, \tag{5}$$
$$P - (1 - 2\lambda)Q + (1 - 2\lambda)^2 R - (1 - 2\lambda)^3 z^3 = 0, \tag{6}$$
$$P - (\lambda + 1)Q + (\lambda + 1)^2 R - (\lambda + 1)^3 z^3 = 0, \tag{7}$$

where, for the sake of brevity, we write

$$\begin{aligned} \omega^3 + 3\omega\omega_1 + \omega_2 + 4M_a\omega &= P, \\ 3\omega^2 z + 3\omega z_1 + 3\omega_1 z + z_2 + 4M_a z &= Q, \\ 3\omega z^2 + 3zz_1 &= R. \end{aligned}$$

In order to simplify (5), (6), and (7), we multiply the first of them by λ, the second by -1, and the third by $1 - \lambda$, and take their sum, which is obviously independent of P, and from which it is easily seen that the terms containing Q and z^3 will also disappear. For

$$\lambda(\lambda - 2) - (1 - 2\lambda) + (1 - \lambda)(\lambda + 1) = 0,$$

and
$$\lambda(\lambda - 2)^3 - (1 - 2\lambda)^3 + (1 - \lambda)(\lambda + 1)^3 = 0.$$

We are thus left with

$$\{\lambda(\lambda - 2)^2 - (1 - 2\lambda)^2 + (1 - \lambda)(\lambda + 1)^2\} R = 0,$$

which, on restoring the value of R and reducing, becomes

$$\lambda(\lambda - 1) z(\omega z + z_1) = 0.$$

Now the values of u, v, w, which are equal to $\frac{X_1}{X}$, $\frac{Y_1}{Y}$, $\frac{Z_1}{Z}$ respectively, being distinct from each other, z cannot vanish; for $z = 0$ would imply $u = v = w$. Hence, considering λ to have any finite numerical value except 1 or 0, we may write
$$\omega z + z_1 = 0$$

in equations (5), (6), (7), which will then become

$$P - (\lambda - 2)(3\omega_1 z + z_2 + 4M_a z) - (\lambda - 2)^3 z^3 = 0, \qquad (8)$$

$$P - (1 - 2\lambda)(3\omega_1 z + z_2 + 4M_a z) - (1 - 2\lambda)^3 z^3 = 0, \qquad (9)$$

$$P - (\lambda + 1)(3\omega_1 z + z_2 + 4M_a z) - (\lambda + 1)^3 z^3 = 0. \qquad (10)$$

Adding these together, we find

$$3P = \{(\lambda - 2)^3 + (1 - 2\lambda)^3 + (\lambda + 1)^3\} z^3$$
$$= 3(\lambda - 2)(1 - 2\lambda)(\lambda + 1) z^3.$$

Restoring the value of P, and writing for shortness

$$(\lambda - 2)(\lambda + 1)(2\lambda - 1) = p,$$

there results
$$\omega^3 + 3\omega\omega_1 + \omega_2 + 4M_a\omega + pz^3 = 0.$$

From any pair of the equations (8), (9), (10) we obtain by subtraction

$$3\omega_1 z + z_2 + 4M_a z + 3(\lambda^2 - \lambda + 1) z^3 = 0.$$

Thus, for example, subtracting (10) from (8), we have

$$3(3\omega_1 z + z_2 + 4M_a z) = \{(\lambda - 2)^3 - (\lambda + 1)^3\} z^3 = -9(\lambda^2 - \lambda + 1) z^3.$$

Collecting our results, we see that equations (5), (6), (7) may be replaced by

$$\omega^3 + 3\omega\omega_1 + \omega_2 + 4M_a\omega + pz^3 = 0, \qquad (11)$$

$$3\omega_1 z + z_2 + 4M_a z + 3qz^3 = 0, \qquad (12)$$

$$\omega z + z_1 = 0, \qquad (13)$$

where $p = (\lambda - 2)(\lambda + 1)(2\lambda - 1)$

and $q = \lambda^2 - \lambda + 1$.

Differentiating (13), we obtain

$$\omega_1 z + \omega z_1 + z_2 = 0.$$

Subtracting this from (12) and adding (13) multiplied by ω, the result divides by z, and we find

$$\omega^2 + 2\omega_1 + 4M_a + 3qz^2 = 0, \qquad (14)$$

which, when multiplied by ω and subtracted from (11), reduces it to

$$\omega\omega_1 + \omega_2 + pz^3 - 3qz^2\omega = 0. \qquad (15)$$

Now it has been shown in Lecture XXX that

$$a^{-\frac{1}{3}} \frac{d}{dx} M_a = 5A_a,$$

$$a^{-\frac{1}{3}} \frac{d}{dx} A_a = 6B_a,$$

$$a^{-\frac{1}{3}} \frac{d}{dx} B_a = 7C_a + M_a A_a,$$

whence it follows that (14) gives on differentiation

$$\omega\omega_1 + \omega_2 + 10A_a + 3qzz_1 = 0.$$

Combining this with (15) we have

$$10A_a = pz^3 - 3qz(\omega z + z_1),$$

or, finally, since $\omega z + z_1 = 0$,

$$10A_a = pz^3.$$

Differentiating this, we have

$$20B_a = pz^2z_1 = -pz^3\omega;$$

i. e.

$$2B_a + A_a\omega = 0, \tag{16}$$

whence, by differentiation,

$$14C_a + 2M_aA_a + 6B_a\omega + A_a\omega_1 = 0.$$

Subtracting (14) multiplied by A_a from the double of this, we have

$$28C_a - A_a\omega^2 + 12B_a\omega - 3qz^2A_a = 0.$$

Substituting in this for ω its value $-\dfrac{2B_a}{A_a}$, found from (16), there results

$$28(A_aC_a - B_a^2) = 3qz^2A_a^2.$$

But it has been shown that

$$10A_a = pz^3.$$

Hence the elimination of z gives

$$28^3p^2(A_aC_a - B_a^2)^3 = 3^3q^3p^2z^6A_a^6 = 10^2 3^3q^3A_a^8.$$

Or restoring for p and q their values in terms of λ, and replacing the absolute reciprocants A_a, B_a, C_a by the non-absolute ones A, B, C (which is effected by merely multiplying throughout by a power of a), we have

$$2^4.7^3(\lambda - 2)^2(\lambda + 1)^2(2\lambda - 1)^2(AC - B^2)^3 = 3^3.5^2(\lambda^2 - \lambda + 1)^3A^8. \tag{17}$$

For other methods of obtaining this differential equation see Halphen's Thèse sur les Invariants Différentiels, p. 30, and Lecture XXX of the present course. It corresponds *in general* (*i. e.* unless $\lambda = 0, 1, \infty$) to the complete primitive

$$Y = X^\lambda Z^{1-\lambda}.$$

When $\lambda = 0, 1, \infty$, the differential equation (17) becomes

$$28^3(AC - B^2)^3 = 3^3.5^2A^8, \tag{18}$$

which corresponds to the complete primitive

$$Y = Xe^{\frac{Z}{X}}. \tag{19}$$

This case has been discussed in the Thèse and in Lecture XXX.

We may obtain (18) from (19) by a method of elimination similar to that employed in deducing (17) from its complete primitive. Thus the first differential of (19) may be written

$$\frac{Y_1}{Y}=\frac{X_1}{X}+\frac{Z_1X-ZX_1}{X^2},$$

which becomes

$$v=u+3z$$

when we assume

$$X_1=Xu,\quad Y_1=Yv,\quad Z_1=Zu+3Xz.$$

By means of the Lemma we obtain

$$u^3+3uu_1+u_2+4M_a u=0, \tag{20}$$

$$v^3+3vv_1+v_2+4M_a v=0, \tag{21}$$

$$3u^2z+3u_1z+3uz_1+z_2+4M_a z=0. \tag{22}$$

The first two of these are identical with (2) and (3) previously given; the third is found as follows. Since

$$\begin{aligned}Z_1&=Zu+3Xz,\\ Z_2&=Z_1u+Zu_1+3X_1z+3Xz_1\\ &=Z(u^2+u_1)+3X(2uz+z_1).\end{aligned}$$

Hence

$$\begin{aligned}Z_3&=Z_1(u^2+u_1)+Z(2uu_1+u_2)+3X_1(2uz+z_1)+3X(2u_1z+2uz_1+z_2)\\ &=Z(u^3+3uu_1+u_2)+3X(3u^2z+3u_1z+3uz_1+z_2).\end{aligned}$$

Thus we have

$$Z_3+4M_aZ_1=Z(u^3+3uu_1+u_2+4M_au)+3X(3u^2z+3u_1z+3uz_1+z_2+4M_az).$$

But $Z_3+4M_aZ_1=0$, and $u^3+3uu_1+u_2+4M_au=0$, which shows that

$$3u^2z+3u_1z+3uz_1+z_2+4M_az=0.$$

Equations (20), (21), and (22), of which we have just proved the last, are merely convenient expressions of the fact that X, Y, Z are linear functions of x, y. We combine them with the first, second, and third differentials of the primitive equation (19) by writing

$$\left.\begin{aligned}v&=u+3z\\ v_1&=u_1+3z_1\\ v_2&=u_2+3z_2\end{aligned}\right\}$$

When this is done (21) becomes

$$(u^3+3uu_1+u_2+4M_au)+3(3u^2z+3uz_1+3u_1z+z_2+4M_az)+27z(uz+z^2+z_1)=0,$$

which, in consequence of the identities (20) and (22), reduces to

$$(u+z)z+z_1=0.$$

Let now $u = \omega - z$, (so that $\omega z + z_1 = 0$). Substituting in (20) and (22) we find

$$\omega^3 + 3\omega\omega_1 + \omega_2 + 4M_a\omega - 3(\omega - z)(\omega z + z_1) - z^3 - 3\omega_1 z - z_2 - 4M_a z = 0,$$

and

$$(3\omega - 6z)(\omega z + z_1) + 3z^3 + 3\omega_1 z + z_2 + 4M_a z = 0$$

respectively. Adding both equations together, and remembering that $\omega z + z_1 = 0$, we obtain

$$\omega^3 + 3\omega\omega_1 + \omega_2 + 4M_a\omega + 2z^3 = 0, \tag{23}$$

$$3\omega_1 z + z_2 + 4M_a z + 3z^3 = 0, \tag{24}$$

which, combined with

$$\omega z + z_1 = 0, \tag{25}$$

replace the system (20), (21), (22).

Comparing these equations with (11), (12), (13), we see that the two sets are identical if we make $\lambda = 0$, when p becomes 2 and $q = 1$. Hence, by performing exactly the same work as in the previous case, we shall find

$$5A_a = z^3 \quad \text{(instead of } 10A_a = pz^3\text{)}$$

and

$$28(A_a C_a - B_a^2) = 3z^2 A_a^2 \quad \text{(instead of } 3qz^2 A_a^2\text{)}.$$

And, finally, eliminating z between this pair of equations, at the same time replacing the absolute reciprocants A_a, B_a, C_a by the corresponding non-absolute ones A, B, C, we have

$$28^3(AC - B^2)^3 = 3^3 . 5^2 A^8,$$

which is what (17) becomes when λ has any of the values 0, 1, or ∞.

Algebraic Surfaces of which every Plane-Section is Unicursal in the Light of n-Dimensional Geometry.

By Eliakim H. Moore, Jr., *Evanston, Ill.*

Picard, in a recent memoir,* established the following theorem :

Les seules surfaces algébriques dont toutes les sections planes sont unicursales, sont les surfaces réglées unicursales et la surface du quatrième degré de Steiner.

In the present article I wish to give another proof of the same theorem, and to develop several allied propositions in the geometry of n-dimensions.

Picard notices at once that a surface of the kind under consideration, viz., of which every plane-section is unicursal, must be itself unicursal, and, accordingly, that there is a 1.1 correspondence between a point of the surface and a point of a plane (determined respectively by the homogeneous coordinate-sets (x, y, t, u), (α, β, γ)) defined by the equations

$$(1) \quad x=f_1(\alpha, \beta, \gamma),\ y=f_2(\alpha, \beta, \gamma),\ z=f_3(\alpha, \beta, \gamma),\ t=f_4(\alpha, \beta, \gamma),$$

where the f are integral homogeneous functions of α, β, γ of, say, degree n.

It is shown, l. c., pp. 77, 78, that there is no loss of generality in assuming that all the multiple points common to the triply-infinite system of curves

$$(2) \quad Af_1(\alpha, \beta, \gamma)+Bf_2(\alpha, \beta, \gamma)+Cf_3(\alpha, \beta, \gamma)+Df_4(\alpha, \beta, \gamma)=0$$

are ordinary multiple points; and that in these points the curves have no common tangents. Let x_k be the number of k-ple points common to the f-curve-system.

To every curve of the system of curves (2) corresponds, in virtue of (1), the plane-section of the surface lying in the corresponding plane of the triply-infinite system of planes

$$(3) \quad Ax+By+Cz+Dt=0.$$

Every plane-section is unicursal; so every curve of system (2) must be unicursal.

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* "Sur les surfaces algébriques dont toutes les sections planes sont unicursales" (Kronecker's Journal d. Math., 1886, C, pp. 71-78).

If N is the degree of the surface, two planes intersect in a straight line which meets the surface in N points; any two plane-sections intersect in N points; so any two curves of the system must intersect in N points (distinct from the common base-point system common to all the curves). In this way we have the intersection- and unicursality-equations

$$(4) \qquad \sum_k k^2 x_k = n^2 - N,$$

$$(5) \qquad \sum_k \frac{1}{2} k(k-1) x_k = \frac{1}{2}(n-1)(n-2),$$

where in (5) the justifiable assumption is made that the arbitrary curve of the system (2) has no multiple point outside of the common base-point system. Subtracting, we have

$$(6) \qquad \sum_k \frac{1}{2} k(k+1) x_k = \frac{1}{2} n(n+3) - (N+1).$$

In order that a point should be a k-ple point on a curve, $\frac{1}{2} k(k+1)$ conditions must be imposed on the curve. A curve of order n is determined by $\frac{1}{2} n(n+3)$ conditions. So from (6) the curves of order n passing through the common base-points of the system (2) contain $N+1$ arbitrary parameters; that is, are determined by $N+2$ linearly independent curves C^n of the system.

Here we cease to follow Picard, and notice that *a surface of order N of which every plane-section is unicursal may be regarded as the projection of a two-dimensional surface of order N in a flat space of $N+1$ dimensions.*

After Clifford, for *two-dimensional surface* we shall say *two-spread*, and for *flat space of $N+1$ dimensions* simply $(N+1)$-*flat.*

Let $y_1, y_2, \ldots y_{N+2}$ be the homogeneous coordinates of a point in an $(N+1)$-flat R_{N+1}. Then the equations

$$(7) \qquad y_\kappa = f_\kappa(\alpha, \beta, \gamma), \qquad (\kappa = 1, 2, \ldots N+2),$$

where $f_\kappa(\alpha, \beta, \gamma) = 0$ for $\kappa = 1, 2, \ldots N+2$ are the equations of any $N+2$ linearly independent curves of the system C^n, of which the first four are, however, the f_1, f_2, f_3, f_4 of the preceding investigation, give the point-point representation on the (α, β, γ)-plane of a certain two-spread in the $(N+1)$-flat R_{N+1}. This two-spread is met by the $(N-1)$-flat of intersection of the two $(N+1)$-flats

$$\sum_\kappa a'_\kappa y_\kappa = 0, \quad \sum_\kappa a''_\kappa y_\kappa = 0$$

in a number of points equal to the order of the two-spread; these points correspond to the N points of intersection of the two curves of the system

$$\sum_\kappa a'_\kappa f_\kappa(\alpha, \beta, \gamma) = 0, \quad \sum_\kappa a''_\kappa f_\kappa(\alpha, \beta, \gamma) = 0:$$

Hence the two-spread is of order N.

The equations (7) show at once that the two-spread may be projected from the $(N-3)$-flat passing through the coordinate vertices $5, 6, \ldots . N+2$ on the 3-flat $y_5 = y_6 = \ldots . = y_{N+2} = 0$ into the surface under consideration,

$$y_1 = f_1(\alpha, \beta, \gamma), \quad y_2 = f_2(\alpha, \beta, \gamma), \quad y_3 = f_3(\alpha, \beta, \gamma), \quad y_4 = f_4(\alpha, \beta, \gamma),$$

if for x, y, z, t of equations (1) we write y_1, y_2, y_3, y_4.

Conversely, a two-spread of order N in an $(N+1)$-flat is unicursal and every N-flat section of it is unicursal. For by successive projections from points on the spread it is projected into a two-spread of order $N-1$ in an N-flat, of order $N-2$ in an $(N-1)$-flat $\ldots .$, of order 2 in a 3-flat and then into a plane. In like manner a curve of order N (in which the two-spread is cut by an N-flat R_N) in an N-flat may be projected into a line.*

Thus a two-spread of order N in an $(N+1)$-flat is itself unicursal and every flat-section of it is unicursal; and clearly, however projected, the spread retains these characteristics; in particular, it is projected from an $(N-3)$-flat not intersecting it upon a three-flat into a unicursal surface of order N every plane-section of which is unicursal.

The 1.1 *Correspondence between a Two-spread of Order N in an $(N+1)$-flat and a Plane.*

$y_1, y_2, \ldots . y_{N+2}$ are the homogeneous point-coordinates in the $(N+1)$-flat R_{N+1}. Choose $N-1$ points, $A_1, A_2, \ldots . A_{N-1}$, of the spread at random; that is, so that the $(N-2)$-flat passing through them meets the spread in no other point. Let

$$L' \equiv \Sigma l'_\kappa y_\kappa = 0, \quad L'' \equiv \Sigma l''_\kappa y_\kappa = 0, \quad L''' \equiv \Sigma l'''_\kappa y_\kappa = 0$$

be three asyzygetic N-flats through these points A. Then

$$(8) \quad \beta L' - \alpha L'' = 0, \qquad (9) \quad \gamma L' - \alpha L''' = 0$$

is an R_{N-1} meeting the spread in N points; that is, in the $N-1$ fixed points A and in one other point P depending on the ratios $\alpha:\beta:\gamma$. In fact, by a suitable choice of coordinate axes of α, β, γ in a fixed plane in the $(N+1)$-flat, the point of intersection P' of the R_{N-1} with this plane will have the homogeneous coordinates α, β, γ. This is the 1.1 correspondence by projection between the points of the two-spread and those of a plane.

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* Clifford: "On the Classification of Loci," Mathematical Papers, pp. 305-331.

Consider the system of equations formed by joining to (8) and (9) the equations of the two-spread (equivalent to $N-1$ independent equations) and the equation of a variable N-flat

$$(10) \qquad u_y \equiv u_1 y_1 + u_2 y_2 + \ldots + u_{N+2} y_{N+2} = 0.$$

From these equations—one more than sufficient to determine $y_1, \ldots y_{N+2}$—we may eliminate $y_1, \ldots y_{N+2}$. The eliminant equated to zero is the condition that the N-flat $u_y = 0$ pass through some one of the N points common to (8), (9) and the spread, is in fact the tangential equation of these N points, and so is of degree N in the u. The eliminant is of degree N in α, β, γ also, because, considering $\alpha : \beta$ and the u as constant, there will be N values of $\alpha : \gamma$ which will make the equations consistent, or, what is the same thing, make the eliminant vanish, viz., the N values for every one of which the N-flat $\gamma L' - \alpha L''' = 0$ passes through one of the N points common to $\beta L' - \alpha L'' = 0$, the spread and the N-flat $u_y = 0$.

The eliminant is then a homogeneous function of the N^{th} degree of the u, and also of α, β, γ. Considered as a function of the u and equated to zero, it is the tangential equation of the N points common to (8), (9) and the spread; *i. e.*, of the $N-1$ fixed points A and of the one variable point P depending upon $\alpha : \beta : \gamma$. Hence the eliminant may be separated into the product of $N-1$ factors linear in the u and independent of α, β, γ (the tangential expressions for the $N-1$ fixed points A), and one factor linear in the u and of degree N in α, β, γ (the tangential expression of the point P). From this last factor we have the tangential equation of P,

$$(11) \qquad u_1 F_1(\alpha, \beta, \gamma) + u_2 F_2(\alpha, \beta, \gamma) + \ldots + u_{N+2} F_{N+2}(\alpha, \beta, \gamma) = 0,$$

which must be identical with

$$(12) \qquad u_1 y_1 + u_2 y_2 + \ldots + u_{N+2} y_{N+2} = 0,$$

where the y are the coordinates of point P.

Hence the coordinates of a point P of the spread are proportional to homogeneous functions of degree N of the coordinates of P' the projection of P on the (α, β, γ)-plane; that is,

$$(13) \qquad y_\kappa = \rho F_\kappa(\alpha, \beta, \gamma), \qquad (\kappa = 1, 2, \ldots N+2),$$

where ρ is an arbitrary constant, the proportion-factor.

There is a 1.1 correspondence between the points on the *unicursal* curve of intersection of the spread with the N-flat R_N,

(14) $$a_1y_1 + a_2y_2 + \dots + a_{N+2}\,y_{N+2} = 0,$$

and the points of the curve of order N in the correspondence-plane

(15) $$a_1F_1(\alpha, \beta, \gamma) + a_2F_2(\alpha, \beta, \gamma) + \dots + a_{N+2}F_{N+2}(\alpha, \beta, \gamma) = 0.$$

Hence this curve, in fact every curve of the system determined by the $N+2$ fundamental curves $F_\kappa(\alpha, \beta, \gamma) = 0$, must be unicursal. Two curves of this system must meet in and only in N points (aside from the points common to all curves of the system), for these points are to correspond to the N points where the spread is intersected by the R_{N-1} of intersection of the two R_N corresponding to the two curves. In R_{N+1} the system of R_N is $(N+1)$-ply infinite, hence the corresponding system of curves must be $(N+1)$-ply infinite.

Suppose the system of curves of order N has α_r common r-ple points (for the moment assumed to be without further singularity). The intersection- and unicursality-equations may be written

(I) $$\Sigma r^2\alpha_r = N^2 - N,$$

(II) $$\Sigma \frac{1}{2} r(r-1)\,\alpha_r = \frac{1}{2}(N-1)(N-2).$$

The base-point system cannot contain two multiple points the sum of whose orders is greater than N; otherwise *every* curve of the system would *break up* into the line joining the two multiple points and a supplementary curve, which is impossible. Hence, either all the α_k for $k > \frac{N}{2}$ equal 0, or if one equals unity all the remaining ones equal 0.

From (II) we have what is the same thing,

(II) $$\Sigma r(r-1)\,\alpha_r = N^2 - 3N + 2 = (N-1)(N-2),$$

which, subtracted from (I), gives

(III) $$\Sigma r\alpha_r = 2N - 2 = 2(N-1).$$

There is a general, say *the general*, solution,

$$\alpha_{N-1} = 1, \quad \alpha_1 = N-1; \quad \alpha_r = 0 \qquad (N-1 > r > 1).$$

There is no other solution with a multiple point of order $> \frac{N}{2}$, say of order $N-s$, where $1 < s < \frac{N}{2}$. For, suppose there were such a multiple point, the multiple point of next highest order might be of order s (so that sum of orders shall just equal but not exceed N).

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In (II), (III) substitute for a_{N-s} 1, and for α_r $(s<r \neq N-s)$ 0, and expose outside the summation signs the leading terms, and we have

(II*a*)
$$s(s-1)\alpha_s + \underset{t<s,}{\Sigma t(t-1)\alpha_t} = (s-1)(2N-s-2),$$

(III*a*)
$$s\alpha_s + \Sigma t\alpha_t = N+s-2.$$

Multiplying (III*a*) by $s-1$ and subtracting (II*a*), one has

(IV*a*)
$$(s-1)\Sigma t\alpha_t - \Sigma t(t-1)\alpha_t = (s-1)(-N+2s).$$

Now since $s>1$ and also $s>t$, the left side is positive or zero, while since $s<\frac{N}{2}$, the right side is negative. That is, the assumption that there is a solution—in addition to the general solution—with a multiple point of an order $>\frac{N}{2}$ leads to an absurd equation.

Any other solution must then have the order of the multiple point on highest order, say s, $\leqq \frac{1}{2}N$. The equations become, where $t<s$,

(II*b*)
$$s(s-1)\alpha_s + \Sigma t(t-1)\alpha_t = (N-1)(N-2),$$

(III*b*)
$$s\alpha_s + \Sigma t\alpha_t = 2(N-1).$$

Multiplying (III*b*) by $s-1$ and subtracting (II*b*), one has

(IV*b*)
$$(s-1)\Sigma t\alpha_t - \Sigma t(t-1)\alpha_t = (N-1)(2s-N).$$

Here the right side is negative when $s<\frac{1}{2}N$,

or zero when
$$s=\frac{1}{2}N,$$

and not positive, since by hypothesis $s \leqq \frac{1}{2}N$.

The left side $(t<s)$ is positive unless either $s=1$, and $\therefore t=0$, or $\alpha_t=0$ for every $t<s$, when it is zero.

Thus there are only two possibilities: $s=\frac{1}{2}N$ and either $s=1$ or ($s>1$, but) $\alpha_t=0$ for every $t<s$. In the first case,

$$s=1=\frac{1}{2}N, \therefore N=2.$$

From (III*b*) $\quad s\alpha_s=\alpha_1=2(N-1)=2.$

This is, however, in fact, *the general* solution for $N=2$,

$$\alpha_1 \text{ quâ } \alpha_{N-1}=1 \text{ and } \alpha_1 \text{ quâ } \alpha_1=N-1=1; \text{ or } \alpha_1=2.$$

In the second case,

$$s=\frac{1}{2}N \text{ and } \alpha_t=0 \text{ for every } t<s.$$

Equation (IIIb), with which (IIb) in view of (IVb) is identical, becomes

$$s\alpha_s = 2(N-1) = 4s - 2,$$
$$s(4-\alpha_s) = 2.$$

Since $s > 1$, $s = 2$, $\therefore 4 - \alpha_s = 4 - \alpha_2 = 1$, $\therefore \alpha_2 = 3$, and $N = 2s = 4$.
This exceptional solution, $N = 4$, $\alpha_2 = 3$, gives the 1.1 correspondence between an exceptional two-spread of order $N = 4$ in five-flat R_5 and the (α, β, γ)-plane,

(16) $$y_\kappa = \rho F_\kappa(\alpha, \beta, \gamma), \qquad (\kappa = 1, 2, \ldots. 6),$$

where the F are homogeneous functions of α, β, γ of fourth degree, and such that

(17) $$a_1F_1(\alpha, \beta, \gamma) + a_2F_2(\alpha, \beta, \gamma) + \ldots. + a_6F_6(\alpha, \beta, \gamma) = 0$$

is the equation of the 5-ply infinite system of unicursal quartic curves through three double points. By a quadratic transformation in the (α, β, γ)-plane of which the three double points are the fundamental points, this system of quartic curves transforms into the general (5-ply infinite) system of plane conics, say

(18) $$F_\kappa(\alpha, \beta, \gamma) = \rho' G_\kappa(\alpha', \beta', \gamma'), \qquad (\kappa = 1, 2, \ldots. 6),$$

where the G are homogeneous functions of α', β', γ' (the new coordinates) of the second degree, and ρ' is a proportion-factor. Thus the correspondence is given by

(19) $$y_\kappa = \rho'' G_\kappa(\alpha', \beta', \gamma'), \quad (\kappa = 1, 2, \ldots. 6), \ (\rho'' = \rho\rho').$$

I call this spread *Steiner's quartic two-spread in a space of five dimensions.* For it projects by the planes passing through a fixed line on a fixed three-flat into an ordinary Steiner's quartic in three dimensions. The coordinate-system $y_1, y_2, \ldots. y_6$ is as yet perfectly general. We may specialize by taking the coordinate-vertices 5, 6 on the line from which we project, and by taking for $y_5 = 0$ and $y_6 = 0$ two R_4 passing through the R_3 on which we project. Then the two-spread will be projected into a unicursal surface in the R_3 whose point for point correspondence with a plane is given, after suitable choice of planes of reference, by the equations

(20) $$y_\kappa = \rho'' G_\kappa(\alpha', \beta', \gamma'), \qquad (\kappa = 1, 2, 3, 4);$$

or, in fact, Steiner's quartic.

The general solution

$$\alpha_{N-1} = 1, \quad \alpha_1 = N - 1$$

gives the 1.1 correspondence between a two-spread of order N in an $(N+1)$-flat and the (α, β, γ)-plane,

(21) $$y_\kappa = \rho F_\kappa(\alpha, \beta, \gamma), \qquad (\kappa = 1, 2, \ldots. N+2),$$

where the F are homogeneous functions of α, β, γ of degree N, and such that

(22) $$a_1F_1(\alpha, \beta, \gamma) + a_2F_2(\alpha, \beta, \gamma) + \ldots + a_{N+2}F_{N+2}(\alpha, \beta, \gamma) = 0$$

is the equation of the $(N+1)$-ply infinite system of curves of order N through one $(N-1)$-ple point and $N-1$ simple points. A line through the $(N-1)$-ple point is met by a curve of the system in $N-(N-1) = 1$ effective point. The curve on the two-spread corresponding to this line must then itself be a line, being met in only one point by the R_N meeting the two-spread in the curve corresponding to the curve of the plane-system. That is, all two-spreads of order N in an $(N+1)$-flat are *ruled*.

It is now necessary to remove the restriction in singularity made with reference to the common multiple points of the system of curves of order N in the (α, β, γ)-plane. Clebsch (Vorlesungen über Geometrie, pp. 491–496) has shown how an i-ple point, as complicated as may be, may be considered as an i-ple point which has absorbed a certain number of other multiple points of definite order; of these, every k-ple point is equivalent (in questions relating to the class or the deficiency of the curve) to $\frac{1}{2}k(k-1)$ double-points of which a certain number are cusps. Now, from this standpoint, let γ_r denote the number of r-ple points common to all curves of the system (that is, $\gamma_r = \alpha_r + \beta_r$, where α_r is the number of explicit r-ple points common and β_r is the number of r-ple points absorbed by the various explicit multiple points common to all curves of the system); and let c denote the number of cusps. In the unicursality-equation a cusp plays the rôle of an ordinary double-point. In the intersection-equation a k-ple point (*whether explicit or absorbed*), common to the two curves, counts for a number of intersections equal to $k^2 +$ the number of cusps it contains $+$ the number of intersections in multiple points absorbed by it. (Clebsch, to be sure, discusses only one curve, and not at all the intersection of two curves; but the truth of the statement made will appear from the discussion of Clebsch, when one bears in mind that in an ordinary quadratic transformation with a fundamental point at the k-ple point common to the two curves, k^2 of the intersections at the k-ple point disappear, but all others remain as intersections of the transformed curves.) Thus the base-point system is equivalent to $\Sigma r^2\gamma_r + c$ intersections of two arbitrary curves of the system.

The general intersection- and unicursality-equations are then

(I) $$\Sigma r^2\gamma_r + c = N^2 - N,$$

(II) $$\Sigma \frac{1}{2} r(r-1)\gamma_r = \frac{1}{2}(N-1)(N-2),$$

whence

(III) $$\Sigma r\gamma_r + c = 2N - 2.$$

The general solution is

$$\gamma_{N-1} = 1, \quad \gamma_1 + c = N - 1.$$

The base-point system cannot contain two multiple points (explicit or absorbed) the sum of whose orders exceeds N; for if so, every curve of the system would degenerate, which is not admissible. This will be proved presently; but with its aid we can show, as before, that aside from the general solution just given, there is the single exceptional solution corresponding to Steiner's quartic two-spread,

$$c = 0, \quad N = 4, \quad \gamma_2 = 3.$$

The equations (IV*a*), (IV*b*) are in this case, where the s and t have the same meanings as before,

(IV*a*) $$c(s-1) + (s-1)\Sigma t\gamma_t - \Sigma t(t-1)\gamma_t = (s-1)(-N+2s),$$

which is, as before, an absurdity;

(IV*b*) $$c(s-1) + (s-1)\Sigma t\gamma_t - \Sigma t(t-1)\gamma_t = (N-1)(2s-N),$$

which furnishes two possibilities: first,

$$s = 1, \quad N = 2, \quad c = 0,$$

since only multiple points contain cusps, $\gamma_1 = 2$, which is a particular case of the general solution; second,

$$s > 1, \quad s = \frac{1}{2}N, \quad c = 0, \quad s = 2, \quad N = 4, \quad \gamma_2 = 3,$$

which corresponds to a Steiner's quartic two-spread as previously defined, as will be shown later.

We return to the proof that if the sum of the orders of two multiple points (explicit or absorbed) on a curve of order N exceeds N, the curve degenerates. The order of an absorbed multiple point is not greater than that of the absorbing multiple point. The theorem is evident, then, except in the case where an explicit multiple point, say of order i, absorbs an l_1-ple point where $i + l_1 > N$. By reference to Clebsch (Vorlesungen ü. Geom., p. 493), whose notation we use, such an absorption occurs as follows:

Let the i-ple point be at $x_1 = x_2 = 0$, where l tangents $(l \leqq i)$ fall together in the line $k_1 x_1 + k_2 x_2 = 0$, say $\alpha = 0$. The equation of the curve may be written

$$C \equiv g_{i-l}(x_1, x_2)\, \alpha^l x_3^{N-i} + f_{i+1}(x_1, x_2)\, x_3^{N-i-1} + \ldots + f_{i+t}(x_1, x_2)\, x_3^{N-i-t} \\ + \ldots + f_N(x_1, x_2) = 0;$$

where the f, g denote homogeneous functions of the arguments of degree equal to the subscript-numeral.

Now let $f_{i+r}(x_1, x_2)$ for $r = 1 \ldots l_1$ (where $l_1 \leqq l \leqq i$) contain α as a factor at least $l_1 - r$ times. Then the equation may be written

$$C \equiv g_{i-l}(x_1, x_2)\, \alpha^l x_3^{N-i} + \alpha^{l_1-1} g_{i-l_1+2}(x_1, x_2)\, x_3^{N-i-1} \\ + \ldots + \alpha^{l_1-r} g_{i-l_1+2r}(x_1, x_2)\, x_3^{N-i-r} + \text{etc.} = 0.$$

This i-ple point (cf. Clebsch, l. c.) has absorbed an l_1-ple point (and also, with which we are not at present concerned, $l - l_1$ cusps). Now, the hypothesis of our theorem is that

$$i + l_1 > N, \quad l_1 > N - i,$$

and the last term on the left in the equation of the curve is, for $r = N - i$ (in fact $< l_1$),

$$\alpha^{l_1-(N-i)} g_{i-l_1+2(N-i)}(x_1, x_2) = \alpha^{l_1+i-N} g_{2N-l_1-i}(x_1, x_2),$$

and in fact the curve breaks up, consisting of the line α taken $l_1 + i - N$ times and a supplementary curve.

Corresponding to the general solution

$$\gamma_{N-1} = 1, \quad \gamma_1 + c = N - 1,$$

there is, as before, a unicursal ruled two-spread of order N in $(N+1)$-flat.

The exceptional solution

$$N = 4, \quad \gamma_2 = \alpha_2 + \beta_2 = 3$$

has three cases.

(A). $\alpha_2 = 3$, $\beta_2 = 0$. Three explicit double points. This is the case previously discussed which led to the definition of Steiner's quartic two-spread in a five-flat.

(B). $\alpha_2 = 2$, $\beta_2 = 1$. Two explicit double points, say (x_1, x_2), (x_2, x_3), of which one, say (x_1, x_2), has absorbed a third double point along the line x_1. The equation of the system of quartic curves is

$$C \equiv a x_1^2 x_3^2 + x_1 x_2 x_3 (b_1 x_1 + b_2 x_2) + x_2^2 (c_1 x_1^2 + c_2 x_1 x_2 + c_3 x_2^2) = 0.$$

By the quadratic transformation,

$$\left.\begin{aligned} x_1 : x_2 : x_3 &= z_2^2 : z_1 z_2 : z_1 z_3 \\ z_1 : z_2 : z_3 &= x_2^2 : x_1 x_2 : x_1 x_3 \end{aligned}\right\}$$

which has a fundamental point at (x_2, x_3) and two coincident ones at (x_1, x_2) along the line x_1, this system of quartic curves transforms into

$$C' \equiv a z_3^2 + b_1 z_2 z_3 + b_2 z_1 z_3 + c_1 z_2^2 + c_2 z_1 z_2 + c_3 z_1^2;$$

that is, into the 5-ply infinite system of plane conics. Hence the two-spread given by case (B) is a Steiner's quartic two-spread as defined under case (A).

(C). $\alpha_2 = 1$, $\beta_2 = 2$. One explicit double point (x_1, x_2) which absorbs a second along the line x_2, which has in turn absorbed a third (along the conic $x_2 x_3 - x_1^2$). Endeavoring to construct the system of quartic curves having in common a singular point of this nature—after the discussion of Clebsch, a clear though rather long problem—we find that, by suitable choice of reference lines and constants, the equation may be written

$$C \equiv a(x_2 x_3 - x_1^2)^2 - b x_1 x_2 (x_2 x_3 - x_1)^2 + x_2^2 (c x_1^2 + d x_1 x_2 + e x_2^2 + f x_2 x_3) = 0.$$

By the quadratic transformation,

$$\left.\begin{aligned} x_1 : x_2 : x_3 &= z_1 z_2 : z_2^2 : z_2 z_3 + z_1^2 \\ z_1 : z_2 : z_3 &= x_1 x_2 : x_2^2 : x_2 x_3 - x_1^2 \end{aligned}\right\}$$

which has three consecutive fundamental points at (x_1, x_2) along the conic $x_2 x_3 - x_1^2$, this system of quartic curves transforms into

$$C' \equiv a z_3^2 - b z_1 z_3 + c z_1^2 + d z_1 z_2 + e z_2^2 + f(z_2 z_3 + z_1^2) = 0;$$

that is, since the six quadratic expressions are asyzygetic, into the 5-ply infinite system of plane conics. Hence the two-spread given by case (C) is a Steiner's quartic two-spread as originally defined.

The principal results of the article may be collected in the following theorems:

An algebraic two-spread [two-dimensional surface] *of order N in a flat space of any number of dimensions of which every flat section is unicursal, is the projection of a two-spread of order N in an (N + 1)-flat* [flat space of $N + 1$ dimensions].

A two-spread of order N in an (N + 1)-flat is unicursal and every N-flat section of it is unicursal.

All such two-spreads, with the exception of Steiner's quartic two-spread in a five-flat, are ruled.

Projections of these spreads have corresponding properties.

In a memoir by Veronese, "Behandlung der projectivischen Verhältnisse der Räume von verschiedenen Dimensionen durch das Princip des Projicirens und Schneidens" (Math. Annalen, 1882, XIX, pp. 161–234), in V, §4, p. 224, *unicursal ruled* two-spreads of order N in an $(N+1)$-flat are considered, and further by the writer, "Extensions of certain Theorems of Clifford and of Cayley in the Geometry of n-Dimensions"[*] (Trans. Conn. Acad., VII, 1885), who proved there that *all* two-spreads of order N in an $(N+1)$-flat are *unicursal*, and in this article that all, with the exception of Steiner's quartic, are *ruled*.

EVANSTON, ILL., July 2, 1887.

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[*] I take this opportunity of saying that when the article "Extensions," etc., was written I had not seen the article of Veronese, and that my theorems A of I, p. 10, and 1 of IV, p. 24, are given by Veronese on p. 167 and p. 192 respectively, and further in a note, p. 228, he gives my first abbildung-system of p. 12.

On Professor Cayley's Extension of Arbogast's Method of Derivations.

By Morgan Jenkins, M. A.

Professor Cayley's Extension of Arbogast's Method of Derivations (Transactions of the Royal Society, received October 18, read December 13, 1860) gives a method of formation and arrangement of all the combinations of a given degree and weight in the letters $a, b, c, d, \ldots$, the weights of the successive letters being 0, 1, 2, 3, etc., by derivation from one term, namely, the most expanded term of the system. The principle of the arrangement is that the smallest change in the index of the power of any letter prevails over any number of changes in the indices of the powers of any subsequent letters; that is, any term containing c^3 will precede any term containing c^2 accompanied by the same, or by lower powers of the preceding letters, irrespective of any changes which may occur in the subsequent letters. If we adopt Professor Cayley's comparison of this arrangement to a genealogical tree of descent in tail male, we may perhaps call it the genealogical arrangement.

I propose to make some remarks with a view to facilitate Professor Cayley's process; but, as it would still be subject to the difficulty referred to in the last two lines of his memoir, namely, "that a considerable amount of practice would be required before the process could be readily made use of," I propose to explain another process which will, I hope, be considered to have some advantage in respect of facility and certainty of formation, apart from the saving of time and space arising from the use of associated collections.

Following Professor Cayley, we describe the final distinct letters of the term considered as the ultimate letter, the penultimate letter, the antepenultimate letter, and the proantepenultimate letter; the words ultimate, penultimate, antepenultimate, and proantepenultimate denoting the powers of those letters which exist in the given term. The last letter, the last but one, etc., denote the final letters of the series from which the letters of the term are taken.

Arbogast's method of derivation consists in advancing a letter of the ultimate one place, if possible, that is, if it is not the last letter, and also a letter of the penultimate one place, provided it is the contiguous letter or the immediate antecessor in the alphabet of the ultimate letter: thus, if h be the last letter,

from df^2g^2 we derive df^2gh and dfg^3,
from deg^3 " " only deg^2h,
from d^2g^2h " " only d^2gh^2.

This process is a process of dragging by means of which from one term we may obtain either one or two terms of one greater weight.

A corresponding process may be applied to the beginning of a term, the words primary, secondary, ... , first, second, ... taking the place of ultimate, penultimate, ... , last, last but one, etc., respectively; thus,

from df^2g^2 we should derive only cf^2g^2,
from deg^3 " " " ceg^3 and d^2g^3,
from d^2g^2h " " " only cdg^2h,

the terms so derived being of one less weight than the terms from which they are respectively derived.

The reverse of the dragging process will be a pushing process; but whilst the constituents which are affected by the dragging process are the not-last ultimate and the contiguous penultimate, the constituents affected by the pushing process are the last-ultimate with the simple penultimate or the simple not-last ultimate alone, suitable changes being made in the names of the constituents affected if we operate on the beginning instead of the end of a term.

If we have a combination of powers of arranged letters divided into any two collections, as $bc^2d^4 \ldots .][g^3h^2i^5 \ldots .$; and we call an end of either collection which is also an end of the whole combination an outside end, and the other ends of the collections inside ends; then, if we either drag the two inside ends or push the two outside ends or push the outside end of one collection, and drag the inside end of the other collection, we obtain a contraction; if we reverse the pushing and dragging we shall obtain an expansion. The effect either of a contraction or of an expansion is to produce a fresh combination having the total weight unaltered, because the weight of one collection is increased by one, and the weight of the other collection is decreased by one. In order to avoid the complication which would arise from the transference of letters from one collection, we may make the restriction that if the extreme letters of the inside ends of the two collections are contiguous letters of the alphabet, they may be treated as final letters of their collections, and, therefore, not movable out of their collections; thus, from $b^5c][d^2f^2g$, considered as a combination of two collections between a and c and between d and h respectively, we may, by pushing the two outside ends, or, what is equivalent, by contracting b with g, obtain $b^4c^2][d^2f^3$; by dragging the two outside ends we obtain $\begin{bmatrix} ab^4c \\ b^6 \end{bmatrix}$—$\begin{bmatrix} d^2f^2h \\ d^2fg^2 \end{bmatrix}$, the meaning of the connection being that either one of one collection may be associated with either one of the other collection. Again, by pushing the inside end of the right-hand collection, instead of dragging the outside end, we may add $defg^2$ to the derived right-hand collection. But we do not, at any rate at present, suppose c or d to be pushed or dragged out of its own collection.

In Professor Cayley's memoir, U, P, A, and P' are used to denote the ultimate, the penultimate, the antepenultimate, and the proantepenultimate respectively; and the rules for the application of the contractions known as reversible contractions are given in the following table, transcribed from the memoir:

	P	A	P'
U	U, P, non-contiguous letters. The ultimate a simple letter, or a power of the last letter.	P, A, contiguous letters. The ultimate a simple letter, or a power of the last letter.	
P		P, A, non-contiguous letters. Penultimate a simple letter. Ultimate the last letter, or a power thereof.	A, P', contiguous letters. Penultimate a simple letter. Ultimate the last letter, or a power thereof.

This table will be found equivalent to a combination of the pushing and dragging rules when the given term is divided into two collections in two different ways, first by cutting off U, and then by cutting off P, U from the given term; if we then push U or P, U from the outside end, and drag .... $P'AP$ or $P'A$ from the inside end, attending to the restriction above stated in regard to inside ends, we shall obtain the rules of the table.

For, from .... $AP][U$ we shall obtain the contraction UP if U, P satisfy the conditions of the left-hand column of the table. From .... $P'A][PU$ we shall obtain the contraction UA if U, P and A satisfy the conditions of the upper portion of the middle column of the table; and so on through the other two cases of the table.

No proof of the rules is offered because they are proved in the memoir from which the table is taken.

The chief difficulty in practice, in forming the genealogical arrangement by means of reversible contractions, will probably be found in correctly applying the general direction which follows the table; that is to say, "The contractions" (not more than 3 in number, because the conditions for the contractions UA, PA are mutually exclusive) "are to be applied in the order UP, UA, PA, PP'," but all the contractions originating in a prior contraction of a given term are to be exhausted before proceeding to a posterior contraction of the same term."

The last part of this direction will be found equivalent to the statement that where the last part of a combination is incontractible in consequence of it containing only powers of two contiguous letters, or less than this as a limiting case, we must go back to the earliest preceding term which contains the previous part of this combination unaltered, and contract the last letter of this previous part with the nearest non-contiguous letter.

Examples may be taken from the following complete genealogical arrangement of combinations of letters having total weight equal to 17, degree equal to 6, and extent equal to 7.

a^3dh^2
egh
f^2h
fg^2
a^2bch^2
a^2bdgh
efh
eg^2
f^2g
a^2c^2gh
$cdfh$
cdg^2
ce^2h
$cefg$
cf^3
a^2d^2eh
d^2fg
de^2g
def^2
a^2e^3f

ab^3h^2
ab^2cgh
dfh
dg^2
e^2h
efg
ab^2f^3
abc^2fh
g^2
$abcdeh$
fg
$abce^2g$
ef^2
abd^3h
d^2eg
d^2f^2
$abde^2f$
abe^4
ac^3eh
fg

ac^2d^2h
deg
df^2
ac^2e^2f
acd^3g
d^2ef
de^3
ad^4f
ad^3e^2
b^4gh
b^3cfh
g^2
b^3deh
fg
b^3e^2g
ef^2
b^2c^2eh
fg
b^2cd^2h
deg
df^2

b^2ce^2f
b^2d^3g
d^2ef
de^3
bc^3dh
eg
f^2
bc^2d^2g
def
e^3
bcd^3f
d^2e^2
bd^4e
c^5h
c^4dg
ef
c^3d^2f
de^2
c^2d^3e
cd^5

If we take the term a^2e^3f, e^3f cannot be contracted, the earliest preceding term containing a^2 is a^2bch^2, from which, by contraction of a with c, we obtain ab^3h^2, which is the next term; or again, if we take the term ab^2f^3, f^3 cannot be contracted, being a limiting case of a sequence in powers of e and f, the earliest preceding term containing ab^2 is ab^2cgh, from which, by contraction of b with g, we obtain abc^2fh, which is the next term.

The contractions made according to the rules of the table are called reversible because, by expansion of the last two letters exclusive of the last, in the derived terms we revert to the generating term. Thus the individual denoted by a^2cdfh has 3 sons denoted by a^2cdg^2, a^2ce^2h, and a^2d^2eh, because, by the rules of the table, 3 contractions can be made on a^2cdfh; and it is indicated that these 3 are brothers by the fact that, by expansions performed upon g^2, e^2 and de respectively, we revert to the same term a^2cdfh.

But we may not have the preceding terms written out, and may desire to obtain the next succeeding term to a given term without resort to the preceding terms. For this purpose we may either use an arithmetical formula or combine a non-reversible contraction with the greatest amount of expansion of the latter part of the term on to which the earlier letter of the contracted pair is thrown, which is consistent with the non-disturbance of the letters of the former part from which the earlier letter of the contracted pair is taken. To obtain the requisite arithmetical formula we must note that if W be the total weight of a given combination of letters, S the sum of the indices or degree, and E the extent or weight of the last letter when the weight of the first letter is zero, the most contracted term is expressed by the formula

$$I\left(\frac{W}{S}\right)^{S-R\left(\frac{W}{S}\right)}\left\{1+I\left(\frac{W}{S}\right)\right\}^{R\left(\frac{W}{S}\right)},$$

and the most expanded term by the formula

$$0^{I\left(S-\frac{W}{E}\right)}\,R\left(\frac{W}{E}\right)E^{I\left(\frac{W}{E}\right)},$$

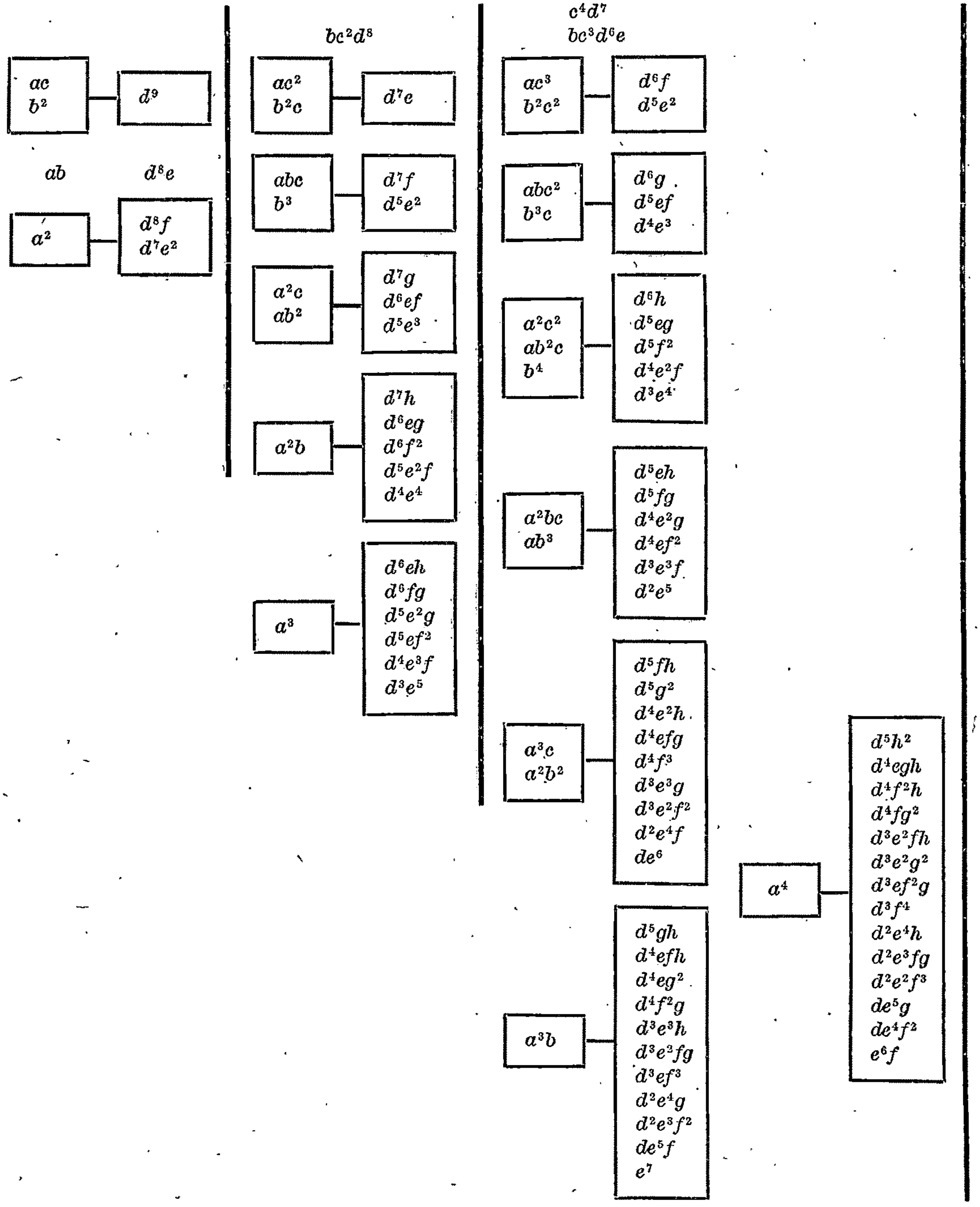
where I and R are used to denote the integral quotient and the remainder in the division expressed by the fraction which follows those letters.

To apply these formulae, suppose we take the term abe^4; here e^4 being incontractible, we wish to expand it to the utmost extent consistent with the non-disturbance of ab. Therefore, taking c for first letter, the weights of e and h are 2 and 5 respectively. Hence, using accented letters for the weight, degree and extent of e^4 when ab is rejected, W' is equal to 8, S' to 4, E' to 5, $I\left(\frac{8}{5}\right)$ to 1, $I\left(4-\frac{8}{5}\right)$ to 2, $R\left(\frac{8}{5}\right)$ to 3. This gives c^2fh as the most expanded term between c and h which is to be derived from e^4. Hence abc^2fh is the earliest term containing ab; and from it, by contracting b with f, we obtain ac^3eh, which is the next term to abe^4.

Or we may expand e^4 to the utmost extent between c and h by any expansions in any order. Thus from e^4 we should obtain in succession de^2f, d^2f^2, c^2g^2, c^2fh, and then proceed as before. It may be noticed that the most contracted term formula shows what is also evident from the method of contraction, that the most contracted term is the same whatever the first and last letters may be, and may therefore be obtained from the formula most conveniently by considering the primary letter to be the first letter and the ultimate letter to be the last letter.

Also, in the formula for finding the most expanded term in general, $I\left(S-\frac{W}{E}\right)$ and $I\left(\frac{W}{E}\right)$ are together equal to $S-1$, and, with the intermediate simple letter of weight $R\left(\frac{W}{E}\right)$, make up the degree S; but if W is a multiple of E, the intermediate simple letter disappears, and $I\left(S-\frac{W}{E}\right)$ and $I\left(\frac{W}{E}\right)$ are together equal to S.

I now go on to another mode of arrangement, illustrated by an example in the form of a table preceding the explanation.


ac
b^2
d^9
ab
d^8e
a^2
d^8f
d^7e^2
bc^2d^8
ac^2
b^2c
d^7e
abc
b^3
d^7f
d^5e^2
a^2c
ab^2
d^7g
d^6ef
d^5e^3
a^2b
d^7h
d^6eg
d^6f^2
d^5e^2f
d^4e^4
a^3
d^6eh
d^6fg
d^5e^2g
d^5ef^2
d^4e^3f
d^3e^5
c^4d^7
bc^3d^6e
ac^3
b^2c^2
d^6f
d^5e^2
abc^2
b^3c
d^6g
d^5ef
d^4e^3
a^2c^2
ab^2c
b^4
d^6h
d^5eg
d^5f^2
d^4e^2f
d^3e^4
a^2bc
ab^3
d^5eh
d^5fg
d^4e^2g
d^4ef^2
d^3e^3f
d^2e^5
a^3c
a^2b^2
d^5fh
d^5g^2
d^4e^2h
d^4efg
d^4f^3
d^3e^3g
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d^2e^4f
de^6
a^3b
d^5gh
d^4efh
d^4eg^2
d^4f^2g
d^3e^3h
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d^2e^4g
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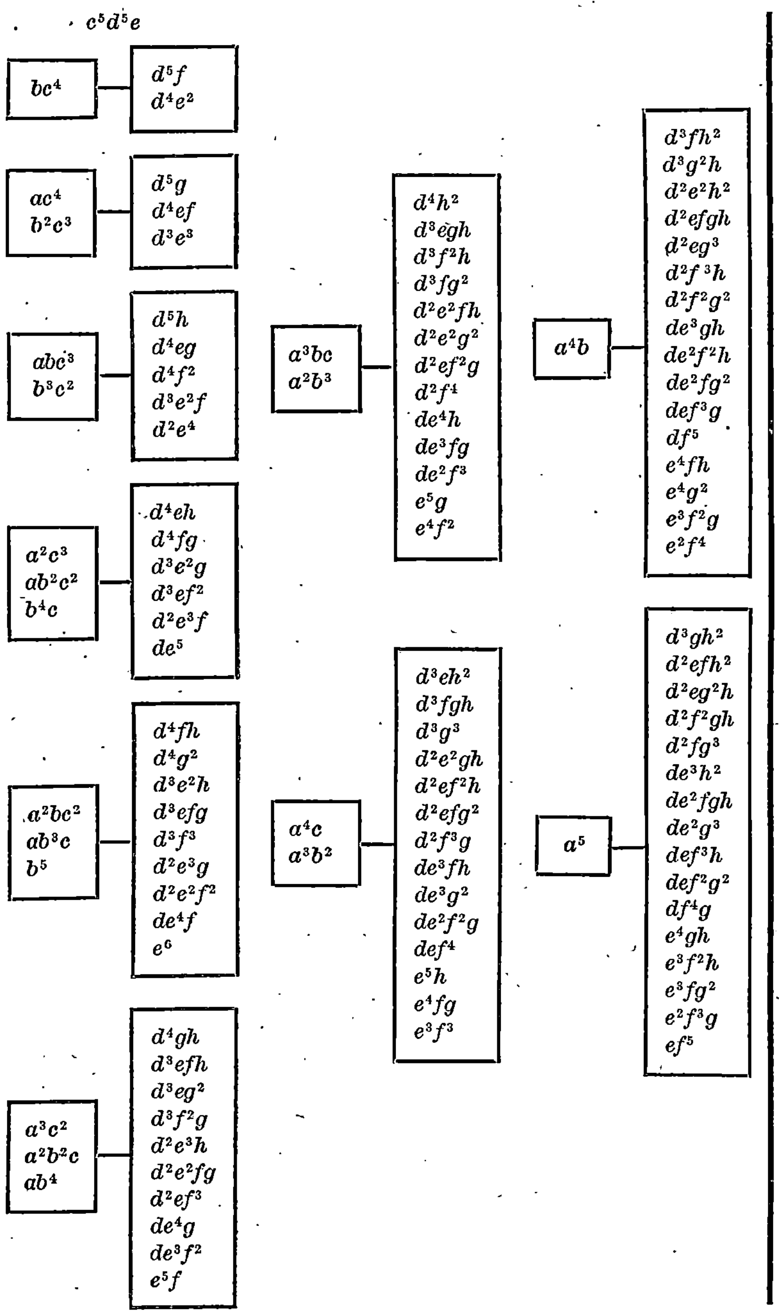
de^5f
e^7
a^4
d^5h^2
d^4egh
d^4f^2h
d^4fg^2
d^3e^2fh
$d^3e^2g^2$
d^3ef^2g
d^3f^4
d^2e^4h
d^2e^3fg
$d^2e^2f^3$
de^5g
de^4f^2
e^6f



c^5d^5e

bc^4	d^5f d^4e^2
ac^4 b^2c^3	d^5g d^4ef d^3e^3
abc^3 b^3c^2	d^5h d^4eg d^4f^2 d^3e^2f d^2e^4
a^2c^3 ab^2c^2 b^4c	d^4eh d^4fg d^3e^2g d^3ef^2 d^2e^3f de^5
a^2bc^2 ab^3c b^5	d^4fh d^4g^2 d^3e^2h d^3efg d^3f^3 d^2e^3g $d^2e^2f^2$ de^4f e^6
a^3c^2 a^2b^2c ab^4	d^4gh d^3efh d^3eg^2 d^3f^2g d^2e^3h d^2e^2fg d^2ef^3 de^4g de^3f^2 e^5f
a^3bc a^2b^3	d^4h^2 d^3egh d^3f^2h d^3fg^2 d^2e^2fh $d^2e^2g^2$ d^2ef^2g d^2f^4 de^4h de^3fg de^2f^3 e^5g e^4f^2
a^4c a^3b^2	d^3eh^2 d^3fgh d^3g^3 d^2e^2gh d^2ef^2h d^2efg^2 d^2f^3g de^3fh de^3g^2 de^2f^2g def^4 e^5h e^4fg e^3f^3
a^4b	d^3fh^2 d^3g^2h $d^2e^2h^2$ d^2efgh d^2eg^3 d^2f^3h $d^2f^2g^2$ de^3gh de^2f^2h de^2fg^2 def^3g df^5 e^4fh e^4g^2 e^3f^2g e^2f^4
a^5	d^3gh^2 d^2efh^2 d^2eg^2h d^2f^2gh d^2fg^3 de^3h^2 de^2fgh de^2g^3 def^3h def^2g^2 df^4g e^4gh e^3f^2h e^3fg^2 e^2f^3g ef^5

c^6	d^4f d^3e^2
bc^5	d^4g d^3ef d^2e^3
ac^5 b^2c^4	d^4h d^3eg d^3f^2 d^2e^2f de^4
abc^4 b^3c^3	d^3eh d^3fg d^2e^2g d^2ef^2 de^3f e^5
a^2c^4 ab^2c^3 b^4c^2	d^3fh d^3g^2 d^2e^2h d^2efg d^2f^3 de^3g de^2f^2 e^4f
a^2bc^3 ab^3c^2 b^5c	d^3gh d^2efh d^2eg^2 d^2f^2g de^3h de^2fg def^3 e^4g e^3f^2
a^3c^3 $a^2b^2c^2$ ab^4c b^6	d^3h^2 d^2egh d^2f^2h d^2fg^2 de^2fh de^2g^2 def^2g df^4 e^4h e^3fg e^2f^3
a^3bc^2 a^2b^3c ab^5	d^2eh^2 d^2fgh d^2g^3 de^2gh def^2h $defg^2$ df^3g e^3fh e^3g^2 e^2f^2g ef^4
a^4c^2 a^3b^2c a^2b^4	d^2fh^2 d^2g^2h de^2h^2 $defgh$ deg^3 df^3h df^2g^2 e^3gh e^2f^2h e^2fg^2 ef^3g f^5
a^4bc a^3b^3	d^2gh^2 $defh^2$ deg^2h df^2gh dfg^3 e^3h^2 e^2fgh e^2g^3 ef^3h ef^2g^2 f^4g
a^5c a^4b^2	d^2h^3 $degh^2$ df^2h^2 dfg^2h dg^4 e^3fh^2 e^2g^2h ef^2gh efg^3 f^4h f^3g^2
a^5b	deh^3 $dfgh^2$ dg^3h e^2gh^2 ef^2h^2 efg^2h eg^4 f^3gh f^2g^3
a^6	dfh^3 dg^2h^2 e^2h^3 $efgh^2$ eg^3h f^3h^2 f^2g^2h fg^4

c^7	d^3g d^2ef de^3
bc^6	d^3h d^2eg d^2f^2 de^2f e^4
ac^6 b^2c^5	d^2eh d^2fg de^2g def^2 e^3f
abc^5 b^3c^4	d^2fh d^2g^2 de^2h $defg$ df^3 e^3g e^2f^2
a^2c^5 ab^2c^4 b^4c^3	d^2gh $defh$ deg^2 df^2g e^3h e^2fg ef^3
a^2bc^4 ab^3c^3 b^5c^2	d^2h^2 $degh$ df^2h dfg^2 e^2fh e^2g^2 ef^2g f^4

a^3c^4 $a^2b^2c^3$ ab^4c^2 b^6c	deh^2 $dfgh$ dg^3 e^2gh ef^2h efg^2 f^3g
a^3bc^3 $a^2b^3c^2$ ab^5c b^7	dfh^2 dg^2h e^2h^2 $efgh$ eg^3 f^3h f^2g^2
a^4c^3 $a^3b^2c^2$ a^2b^4c ab^7	dgh^2 efh^2 eg^2h f^2gh fg^3
a^4bc^2 a^3b^3c a^2b^5	dh^3 egh^2 f^2h^2 fg^2h g^4
a^5c^2 a^4b^2c a^3b^4	eh^3 fgh^2 g^3h
a^5bc a^4b^3	fh^3 g^2h^2
a^6c a^5b^2	gh^3
a^6b	h^4

c^8	d^2h deg df^2 e^2f
bc^7	deh dfg e^2g ef^2
ac^7 b^2c^6	dfh dg^2 e^2h efg f^3
abc^6 b^3c^5	dgh efh eg^2 f^2g
a^2c^6 ab^2c^5 b^4c^4	dh^2 egh f^2h fg^2
a^2bc^5 ab^3c^4 b^5c^3	eh^2 fgh g^3
a^3c^5 $a^2b^2c^4$ ab^4c^3 b^6c^2	fh^2 g^2h
a^3bc^4 $a^2b^3c^3$ ab^5c^2 b^7c	gh^2
a^4c^4 $a^3b^2c^3$ $a^2b^4c^2$ ab^6c b^8	h^3

c^9	eh fg
bc^8	fh g^2
ac^8 b^2c^7	gh
abc^7 b^3c^6	h^2

In this table

$$W=29;\ S=11;\ E=7;\ I\left(\frac{29}{11}\right)=2;\ R\left(\frac{29}{11}\right)=7;\ 11-R\left(\frac{29}{11}\right)=4.$$

Hence the most contracted term is c^4d^7.

The system of combinations of different letters of given weight, degree and extent (which is equivalent to a system of partitions, when multipliers are substituted for the indices) is arranged in what is intended to be eight columns; but some of the columns, on account of their length, are broken up, the original columns being separated by divisional lines.

The column first formed is not that placed at the left-hand side, but the one which is headed c^4d^7. This is supposed to be divided into two collections which are considered to be quite distinct. The outside ends of these are dragged simultaneously according to Arbogast's method of derivations, and the derived terms are put down in associated collections, as in the table. All the left-hand collections are between a and c, of degree 4; all the right-hand collections are between d and h, of degree 7. All the collections so formed are complete collections, because they are formed from complete collections by Arbogast's method. c^4 is a complete collection of its own weight and degree between the letters a and c; and, being a power of the highest letter in its own collection, all terms of lower weight having the same degree and extent can be derived from it by Arbogast's method, that is, by dragging towards the lower end; similarly for d^7, interchanging lower and higher, etc. The total weight of the associated collections is unaltered, because at every simultaneous double derivation the weight of one collection is increased, and the weight of the other is decreased by one.

If we started at the other end of the column and had given a^4 as one complete collection, and also all the terms of the associated complete collection between its most expanded term d^5h^2 and its most contracted term e^6f, we could by the reversal of Arbogast's method, that is, by double outside pushing, obtain the whole column in reversed order.

We have next to explain and prove the method of formation of the adjacent columns.

Let $\boxed{W'+1,\ S'}-\boxed{W''-1,\ S''}$

$\boxed{W',\ S'}\vdash\!\dashv\boxed{W'',\ S''}$ represent two associated pairs of complete collections of letters, the capital letters inside the boundary lines expressing the weight and degree of the respective collections, the boundary lines themselves expressing that the collections are complete; and, for brevity, let c be taken for the last letter of the left-hand collection, d for the first letter of the right-hand collection, as in the table. If we remove a letter d from $\boxed{W'', S''}$, we may balance this by adding c to $\boxed{W' + 1,\ S'}$, because $c + 1$ is equivalent to d, and we shall thus have obtained a portion of $\boxed{W' + c + 1,\ S' + 1} - \boxed{W'' - d,\ S'' - 1}$; but only a portion, because, though the right-hand collection is a complete collection, yet the left-hand collection only includes those terms which actually contain c. That the right-hand collection is a complete collection follows from the fact that there is no term of $\boxed{W'' - d,\ S'' - 1}$ which cannot be obtained from some term of $\boxed{W'', S''}$ which does contain d by the removal of a letter d.

To obtain those terms of $\boxed{W' + c + 1,\ S' + 1}$ which do not contain c, we go back to the next higher left-hand collection in the previous column, namely, to $\boxed{W' + 2,\ S'}$, pick the terms in it which do not contain b and add b to them; we thus obtain those terms of $\boxed{W' + c + 1,\ S' + 1}$ or $\boxed{W' + d,\ S' + 1}$ which do not contain c, but which do contain b; there are still required to be found those terms which contain neither b nor c, but which do contain a; for these we take those terms in $\boxed{W' + 3,\ S'}$ which contain neither b nor c and add a to them, and so on. In this way we should obtain an associated pair of complete collections, $\boxed{W' - d,\ S' + 1} - \boxed{W'' + d,\ S'' - 1}$; and from this, by double outside dragging, we could complete the portion of the column in which the left-hand collection has its weight less than $W' - d$, and the right-hand collection has its weight greater than $W'' + d$. The other portion of the column may be obtained by double outside pushing instead of dragging. It will be most convenient to take for $\boxed{W' + 1,\ S'}$ that left-hand collection which has the greatest weight for the degree S'. For, in this case, $\boxed{W' + 2,\ S'}$, etc., having no existence, we need only add letters to one left-hand collection. Thus, in the table, from $\begin{matrix} c^4 & d^7 \\ bc^3 & d^6e \end{matrix}$, by cutting off d from d^6e and attaching c to c^4, we obtain c^5d^5e; and from this, by double outside dragging, the whole of the remainder of the column headed by c^5d^5e. The collections of this column have the degrees 5 and 6 respectively instead of 4 and 7 respectively.

Again, from $\overset{c^5}{\boxed{bc^4}}-\overset{d^5e}{\boxed{\begin{array}{c} d^5f \\ d^4e^2 \end{array}}}$, by removing a letter d from $\boxed{\begin{array}{c} d^5f \\ d^4e^2 \end{array}}$ and adding a letter c to c^5, we obtain $c^6 \boxed{\begin{array}{c} d^4f \\ d^3e^2 \end{array}}$, and from this, as before, we can obtain the whole of the remainder of the column.

Similarly, from the column whose collections are of degrees S' and S'' respectively, we can form the column whose collections are of degrees $S'-1$, $S''+1$ respectively, by removing a letter c from $\boxed{W',\ S'}$ and adding a letter d to $\boxed{W''-1,\ S''}$ with the corresponding additions where necessary. Thus, in the table, from $\begin{matrix} c^4 & d^7 \\ bc^3 & d^6e \end{matrix}$, by removing a letter c from bc^3 and adding a letter d to d^7, we obtain bc^2d^8; and from this, as before, we can obtain the rest of the column whose collections are of degree 3 and 8 respectively.

It may be considered easiest to obtain independently as much as possible of one column from a contiguous column, and then obtain the rest, if necessary, by dragging. Thus, from the bottom pair or expanded end of the column whose collections are of weight 6 and 5 respectively, by removing d from those terms of the right-hand collection which contain d, we obtain $\boxed{\begin{array}{c} fh^3 \\ g^2h^2 \end{array}}$, which is a complete right-hand collection in the next column.

To obtain the associated left-hand collection we add c to a^5b; also b to a^4c^2, the term in the preceding left-hand collection which does not contain c. The next preceding left-hand collection $\boxed{\begin{array}{c} a^4bc \\ a^3b^2 \end{array}}$ has no terms containing neither b nor c. Hence $\boxed{\begin{array}{c} a^5bc \\ a^4b^3 \end{array}}-\boxed{\begin{array}{c} fh^3 \\ g^2h^2 \end{array}}$ is the associated pair which is formed in the next column. All the rest of the column which is towards the top or contracted end may be formed in a similar manner; but the remaining two pairs $\boxed{\begin{array}{c} a^6c \\ a^5b^2 \end{array}}-\boxed{gh^3}$ and $\boxed{a^6b}-\boxed{h^4}$ must be obtained by dragging.

Properties of a Complete Table of Symmetric Functions.

BY CAPTAIN P. A. MACMAHON, R. A., *Royal Military Academy, Woolwich.*

§1. In Vol. V, *American Journal of Mathematics*, Mr. Durfee has set forth the only complete and perfectly arranged table of symmetric functions in existence.

I propose to establish some remarkable features of such a tabulation which, so far as my knowledge extends, have not as yet been noticed.

If we represent the symmetric functions by partitions in ( ) and the literal products by partitions in ( )′, Mr. Durfee's table exhibits each partition ( ) in terms of the partitions ( )′, and inversely each partition ( )′ in terms of the partitions ( ); say these constitute the first and second portions of the table; the secondary diagonal of the square is mainly composed of units, in the exceptional cases a zero replacing a unit. The first and second portions of the table lie, in the main, above and below the secondary diagonal; the fact that this is not invariably the case being entirely due to the peculiar properties of the self-conjugate partitions; in both the 12^{ic} and 13^{ic} tables there are three such, with the consequence that the corresponding portion of the secondary diagonal becomes twisted about its middle point into coincidence with the principal diagonal of the square.

The terminal units, whether lying in a principal or secondary diagonal of the square, are common to both portions of the table in such wise that any unit, together with the numbers in the same row or column lying left of it or above it, belong to the first portion, whilst the same unit, together with the numbers lying in the same row or column to the right of it or below it, belong to the second portion.

It is important to observe that terminal units must lie either in the secondary or principal diagonal, and those in the principal diagonal correspond invariably to self-conjugate partitions.

The number of partitions of the weight being uneven, the number of self-conjugate partitions is also uneven, and in this case one of the terminal units, corresponding to a self-conjugate partition, is at the point of intersection of the two diagonals, the remaining units being symmetrically distributed on the principal diagonal in adjacent squares; on the other hand, if the number of partitions of the weight be even, so also is the number of self-conjugate partitions; and then there is no place for a number at the intersection of the diagonals, the self-conjugate units being now none of them in the secondary diagonal, but symmetrically placed about it in adjacent squares.

§2. In the Quarterly Journal of Pure and Applied Mathematics, No. 85, 1886, I gave the complete statement of the Cayley-Betti law of symmetry; viz. if any two results of the same weight be

$$\sum A_{\lambda^l\mu^m}\ldots(\lambda^l\mu^m\ldots) = \sum A'_{\lambda'^{l'}\mu'^{m'}}\ldots(\lambda'^{l'}\mu'^{m'}\ldots)',$$

$$\sum B_{\lambda'^{l'}\mu'^{m'}}\ldots(\lambda'^{l'}\mu' \quad \ldots) = \sum B'_{\lambda^l\mu^m}\ldots\lambda_l\mu^m\ldots)',$$

then

$$\sum A_{\lambda^l\mu^m}\ldots B'_{\lambda^l\mu^m}\ldots = \sum A'_{\;\;\mu'^{m'}}\ldots B_{\lambda'^{l'}\mu'^{m'}}\ldots$$

I proceed to the application of this theorem.

§3. Regarding the whole square as a matrix of order equal to the number of partitions of the weight, I consider a minor square matrix of any order whose secondary diagonal is coincident with that of the whole matrix; and for clearness I first suppose such a minor matrix to be situated so that its secondary diagonal contains only units, or, what is the same thing, so that it does not intersect or include the square matrix whose principal diagonal consists entirely of units.

Represent any such minor of order s by

$$\begin{array}{cccccccccc}
\alpha_{11} & \alpha_{12} & \alpha_{13} & \ldots & \alpha_{1,s-5} & \alpha_{1,s-4} & \alpha_{1,s-3} & \alpha_{1,s-2} & \alpha_{1,s-1} & 1\,, \\
\alpha_{21} & \alpha_{22} & \alpha_{23} & \ldots & \alpha_{2,s-5} & \alpha_{2,s-4} & \alpha_{2,s-3} & \alpha_{2,s-2} & 1 & a_{2s}, \\
\alpha_{31} & \alpha_{32} & \alpha_{33} & \ldots & \alpha_{3,s-5} & \alpha_{3,s-4} & \alpha_{3,s-3} & 1 & a_{3,s-1} & a_{3s}, \\
\alpha_{41} & \alpha_{42} & \alpha_{43} & \ldots & \alpha_{4,s-5} & \alpha_{4,s-4} & 1 & a_{4,s-2} & a_{4,s-1} & a_{4s}, \\
\alpha_{51} & \alpha_{52} & \alpha_{53} & \ldots & \alpha_{5,s-5} & 1 & a_{5,s-3} & a_{5,s-2} & a_{5,s-1} & a_{5s}, \\
\alpha_{61} & \alpha_{62} & \alpha_{63} & \ldots & 1 & a_{6,s-4} & a_{6,s-3} & a_{6,s-2} & a_{6,s-1} & a_{6s}, \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\alpha_{s-1,1} & 1 & a_{s-1,3} & \ldots & a_{s-1,s-5} & a_{s-1,s-4} & a_{s-1,s-3} & a_{s-1,s-2} & a_{s-1,s-1} & a_{s-1,s}, \\
1 & a_{s2} & a_{s3} & \ldots & a_{s,s-5} & a_{s,s-4} & a_{s,s-3} & a_{s,s-2} & a_{s,s-1} & a_{ss},
\end{array}$$

wherein the α's and the a's belong respectively to the first and second portions of the table.

Applications of the law of symmetry enable us to write down the $s-1$ relations:

$$\begin{aligned}
&\alpha_{1,\,s-1}+a_{2s}=0,\\
&\alpha_{1,\,s-2}+a_{3,\,s-1}\alpha_{1,\,s-1}+a_{3s}=0,\\
&\alpha_{1,\,s-3}+a_{4,\,s-2}\alpha_{1,\,s-2}+a_{4,\,s-1}\alpha_{1,\,s-1}+a_{4s}=0,\\
&\alpha_{1,\,s-4}+a_{5,\,s-3}\alpha_{1,\,s-3}+a_{5,\,s-2}\alpha_{1,\,s-2}+a_{5,\,s-1}\alpha_{1,\,s-1}+a_{5s}=0,\\
&\alpha_{1,\,s-5}+a_{6,\,s-4}\alpha_{1,\,s-4}+a_{6,\,s-3}\alpha_{1,\,s-3}+a_{6,\,s-2}\alpha_{1,\,s-2}+a_{6,\,s-1}\alpha_{1,\,s-1}+a_{6s}=0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\alpha_{12}+a_{s-1,\,3}\alpha_{13}+\dots+a_{s-1,\,s-5}\alpha_{1,\,s-5}+a_{s-1,\,s-4}\alpha_{1,\,s-4}+a_{s-1,\,s-3}\alpha_{1,\,s-3}\\
&\qquad+a_{s-1,\,s-2}\alpha_{1,\,s-2}+a_{s-1,\,s-1}\alpha_{1,\,s-1}+a_{s-1,\,s}=0,\\
&\alpha_{11}+a_{s2}\alpha_{12}+a_{s3}\alpha_{13}+\dots+a_{s,\,s-5}\alpha_{1,\,s-5}+a_{s,\,s-4}\alpha_{1,\,s-4}+a_{s,\,s-3}\alpha_{1,\,s-3}\\
&\qquad+a_{s,\,s-2}\alpha_{1,\,s-2}+a_{s,\,s-1}\alpha_{1,\,s-1}+a_{ss}=0,
\end{aligned}$$

which are very convenient for purposes of verification. From them is deduced

$$\alpha_{11}=\begin{vmatrix}
-a_{2s} & 0 & 0 & \dots 0 & 0 & 0 & 0 & 1\\
-a_{3s} & 0 & 0 & \dots 0 & 0 & 0 & 1 & a_{3,\,s-1}\\
-a_{4s} & 0 & 0 & \dots 0 & 0 & 1 & a_{4,\,s-2} & a_{4,\,s-1}\\
-a_{5s} & 0 & 0 & \dots 0 & 1 & a_{5,\,s-3} & a_{5,\,s-2} & a_{5,\,s-1}\\
-a_{6s} & 0 & 0 & \dots 1 & a_{6,\,s-4} & a_{6,\,s-3} & a_{6,\,s-2} & a_{6,\,s-1}\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
-a_{s-1,\,s} & 1 & a_{s-1,\,3} & \dots a_{s-1,\,s-5} & a_{s-1,\,s-4} & a_{s-1,\,s-3} & a_{s-1,\,s-2} & a_{s-1,\,s-1}\\
-a_{ss} & a_{s2} & a_{s3} & \dots a_{s,\,s-5} & a_{s,\,s-4} & a_{s,\,s-3} & a_{s,\,s-2} & a_{s,\,s-1}
\end{vmatrix}\div\Delta,$$

where Δ is a determinant of order $s-1$ whose secondary diagonal consists entirely of units and having nothing but zeros above this diagonal; so that

$$\Delta=(-)^{s+s-1+s-2+\dots+2}=(-)^{\frac{1}{2}s(s+1)-1};$$

and in the numerator, if we perform on the columns a cyclical substitution, so as to make the first column the last and then change the sign of the last column, we shall obtain a determinant multiplied by $(-)^{s-1}$; whence the sign of the fraction becomes

$$(-)^{s-1+\frac{1}{2}(s)(s+1)-1}=(-)^{\frac{1}{2}s(s-1)},$$

and accordingly

$$\alpha_{11}=(-)^{\frac{1}{2}s(s-1)}\begin{vmatrix}
0 & 0 & 0 & \dots 0 & 0 & 1 & a_{2s}\\
0 & 0 & 0 & \dots 0 & 1 & a_{3,\,s-1} & a_{3s}\\
0 & 0 & 0 & \dots 1 & a_{4,\,s-2} & a_{4,\,s-1} & a_{4s}\\
0 & 0 & 0 & \dots a_{5,\,s-3} & a_{5,\,s-2} & a_{5,\,s-1} & a_{5s}\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
1 & a_{s-1,\,3} & a_{s-1,\,4} & \dots a_{s-1,\,s-3} & a_{s-1,\,s-2} & a_{s-1,\,s-1} & a_{s-1,\,s}\\
a_{s2} & a_{s3} & a_{s4} & \dots a_{s,\,s-3} & a_{s,\,s-2} & a_{s,\,s-1} & a_{ss}
\end{vmatrix};$$

but the matrix of this determinant is, as regards the original matrix, none other than the matrix complementary to α_{11}, in which all the α's are put equal to zero; further, we shall find α_{1j} equal to the product of $(-)^{\frac{1}{2}s(s-1)+1+j}$ and the determinant of its complementary matrix, with the above convention as regards zeros.

Again, consider any α element in the ρ^{th} row and κ^{th} column $\alpha_{\rho\kappa}$, and in conjunction therewith the matrix obtained from the original matrix by deletion of the first $\rho - 1$ rows and the last $\rho - 1$ columns.

The matrix now under view is of order $s - \rho + 1$, and, by the theorem already established, the element considered is the product of $(-)^{\frac{1}{2}(s-\rho+1)(s-\rho)+\kappa+1}$ and the determinant of the matrix which is its complementary with regard to the matrix of order $s - \rho + 1$.

If we now form the complementary matrix of $\alpha_{\rho\kappa}$ with regard to the matrix of order s, we find that its determinant has the same numerical value as before, but that its sign is by the added rows and columns multiplied by

$$(-)^{s+s-1+\ldots\ldots+s-\rho+2} = (-)^{\frac{1}{2}(\rho-1)(2s-\rho+2)};$$

whence the element $\alpha_{\rho\kappa}$ is equal to the determinant of the matrix which is its complement with regard to the matrix of order s multiplied by

$$(-)^{\frac{1}{2}(s-\rho+1)(s-\rho)+\kappa+1+\frac{1}{2}(\rho-1)(2s-\rho+2)},$$

that is, by $(-)^{\frac{1}{2}s(s-1)+\rho+\kappa}$.

Similarly, the α's are expressed in terms of the α's, and we may enunciate the general theorem:

"Given a complete table of symmetric functions arranged on the Durfee system, and isolating any square matrix of order s whose secondary diagonal, being coincident with that of the whole square, consists solely of terminal units, the value of any element belonging to the first or second portions of the table and situated in the ρ^{th} row and κ^{th} column is equal to the product of

$$(-)^{\frac{1}{2}s(s-1)+\rho+\kappa}$$

and the determinant of its complementary matrix, when in such matrix all other elements belonging to the first or second portions respectively are replaced by zero."

The theorem may be otherwise in part exhibited by writing down the identity

$$\left|\begin{array}{llllll}
\alpha_{11} & \alpha_{12} \cdots\cdots \alpha_{1,s-3} & \alpha_{1,s-2} & \alpha_{1,s-1} & 1 \\
0 & 0 \cdots\cdots 0 & 0 & 1 & a_{2s} \\
0 & 0 \cdots\cdots 0 & 1 & a_{3,s-1} & a_{3s} \\
0 & 0 \cdots\cdots 1 & a_{4,s-2} & a_{4,s-1} & a_{4s} \\
\cdots & \cdots\cdots\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots\cdots\cdots & \cdots & \cdots & \cdots \\
0 & 1 \cdots\cdots a_{s-1,s-3} & a_{s-1,s-2} & a_{s-1,s-1} & a_{s-1,s} \\
1 & a_{s2} \cdots\cdots a_{s,s-3} & a_{s,s-2} & a_{s,s-1} & a_{ss}
\end{array}\right|$$

$$= (-)^{\frac{1}{2}s(s-1)}\{\alpha_{11}^2 + \alpha_{12}^2 + \cdots\cdots + \alpha_{1,s-3}^2 + \alpha_{1,s-2}^2 + \alpha_{1,s-1}^2 + 1\},$$

which follows at once from the foregoing results.

§4. We might now investigate a similar theorem for those cases in which the matrix intersects or includes the matrix whose principal diagonal consists entirely of units, and the requisite modification of the determinant rule is easily reached. But, instead of doing this, I will indicate a slight modification of Durfee's arrangement whereby the law above established becomes universally applicable.

It is well known, and moreover very easily proved, that a symmetric function whose partition is self-conjugate contains one, and only one, term whose partition is self-conjugate; the partition of this one term is identical with that of the symmetric function, and, as we know, its coefficient is a terminal unit; it hence follows that the matrix whose principal diagonal consists entirely of terminal units has every other element necessarily zero; as a consequence, if we arrange the self-conjugate partitions of symmetric functions in any order and then place the self-conjugate partitions of the terms in the reverse order, we necessarily confine each portion of the tabulation to half a square and bring all the terminal units into the secondary diagonal.

I therefore propose that Mr. Durfee's arrangement be modified in this manner, so that the self-conjugate partitions are arranged in only a quasi-symmetrical manner. The theorem of §3 will then be applicable to each of the secondary diagonal matrices of the whole matrix.

Royal Military Academy, Woolwich, England, July 8th.

On Binary Sextics with Linear Transformations into Themselves.

BY OSKAR BOLZA.

In the following investigation, which I have undertaken at the suggestion of Prof. F. Klein, I consider those binary sextics which remain unchanged (or are only changed by a constant factor) for certain linear transformations of the variables. In the first section I determine all the binary sextics of this special character; in the second section the relations are established which in these cases exist between the rational absolute invariants of the sextic; and a completion of a theorem of Clebsch's is given, concerning the possibility of transformation of one sextic into another by linear substitution. Finally, in the third section I examine the corresponding relations between the transcendental absolute invariants, that is to say, between the ϑ-moduli τ_{11}, τ_{12}, τ_{22}.

The subject is of interest for the theory of the hyperelliptic modular functions; for if we represent a system of values of the three complex magnitudes τ_{11}, τ_{12}, τ_{22} by means of a point in a space of six dimensions, it can be shown that the aforesaid relations between the τ_{ik} represent the edges and vertices of the fundamental region ("Fundamentalraum") for the hyperelliptic modular functions. And the question, under what conditions a sextic can be transformed into another by linear substitution, is closely connected with the question whether the rational absolute invariants are single-valued ("eindeutig") functions of the variables τ_{11}, τ_{12}, τ_{22}.

SECTION I.

Determination of all the Binary Sextics with Linear Transformations into Themselves.

§1.

I say, for shortness, that a sextic $f(x_1, x_2)$ remains unchanged for the substitution

$$\left.\begin{aligned} x_1 &= px_1' + qx_2' \\ x_2 &= p'x_1' + q'x_2' \end{aligned}\right\} \qquad (S)$$

if
$$f(x_1, x_2) = f'(x'_1, x'_2) = cf(x'_1, x'_2), \tag{1}$$
c being any constant factor.

It is clear that the entire system of the substitutions which leave a given sextic f unchanged, form *a group*, G; we shall show that this group is *finite*, provided the substitutions S be reduced to the determinant 1; that is,

$$pq' - p'q = 1. \tag{2}$$

Proof. The roots α_i of the equation $f(x_1, x_2) = 0$, which we suppose to be all different from each other, are connected with the roots α'_i of the equation $f'(z'_1, z'_2) = 0$ by the relations

$$\alpha'_i = \frac{q'\alpha_i - q}{-p'\alpha_i + p}; \; (i = 1, 2, \ldots . 6); \tag{3}$$

accordingly each root α'_i corresponds to one of the roots α_i. Now, if the substitution S leaves the sextic f unchanged, the roots α'_i must be identical with the α_i, only in different order; therefore, to each substitution S which leaves f unchanged, corresponds a definite permutation,

$$\sigma = \binom{\alpha_i}{\alpha'_i}$$

of the roots α_i, which is determined by the equations (3).

It is easy to see that the permutations σ corresponding to the linear substitutions S likewise form a group, Γ, which is isomorphic with the group G, and to each substitution of G corresponds a single permutation of Γ.

Conversely, to the permutation 1 of the group Γ correspond all the substitutions of G which satisfy the conditions

$$\alpha'_i = \alpha_i,$$

that is,
$$\alpha_i = \frac{q'\alpha_i - q}{-p'\alpha_i + p},$$
or,
$$-p'\alpha_i^2 + (p - q')\alpha_i + q = 0,$$
for $i = 1, 2 \ldots . 6$.

This equation, considered as an equation with the unknown α_i, can be satisfied in two different ways:

1). The coefficients may not be all $= 0$. In this case the equation $f = 0$ has only two distinct roots, and f is reducible by linear substitution to the form

$$f = z_1^{\lambda_1} z_2^{\lambda_2}, \; (\lambda_1 + \lambda_2 = 6);$$

here the group of substitutions which leave f unchanged is evidently infinite.

2). All the coefficients may be $= 0$; that is,

$$p' = 0,\ p = q',\ q = 0;$$

therefore, because of (2),

$$p = +1,\ q' = +1,\ \text{or}\ p = -1,\ q' = -1.$$

That is to say, to the permutation 1 in the group Γ correspond in the group G the two substitutions

$$x_1 = +x_1',\ x_2 = +x_2',\ \text{and}\ x_1 = -x_1',\ x_2 = -x_2'.$$

Excepting the trivial case where $f = z_1^{\lambda_1} z_2^{\lambda_2}$, we have therefore the result:

The group Γ, which contains certainly only a finite number of distinct operations, is hemiedrically isomorphic with the group G of homogeneous substitutions of the determinant 1, and it is holoedrically isomorphic with the corresponding group of non-homogeneous* substitutions. Thence follows immediately that the group G is a finite group, which was to be proved.

§2.

The group G being a finite group of binary substitutions, must be one of the following well-known groups:†

1). The self-evident group $x_1 = \pm x_1'$, $x_2 = \pm x_2'$.
2). A cyclic group.
3). A diedron group.
4). The tetraedron group.
5). The oktaedron group.
6). The ikosaedron group.

The group $x_1 = \pm x_1'$, $x_2 = \pm x_2'$ leaves *every* binary sextic unchanged; to each of the other groups belong three canonical invariant groundforms; denoting them in the notation of Klein by F_1, F_2, F_3, any binary quantic which remains unchanged for the group in question is reducible by linear transformation to the form‡

$$F_1^{\alpha} F_2^{\beta} F_3^{\gamma} \Pi_i \left(A^{(i)} F_1^{\nu_1} + B^{(i)} F_2^{\nu_2}\right), \tag{4}$$

the $A^{(i)}$, $B^{(i)}$ being any constant factors; the α, β, γ, ν_1, ν_2 being integers.

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* Concerning the difference between homogeneous and non-homogeneous substitutions cf. F. Klein, Vorlesungen über das Ikosaeder, p. 44.

† Cf. F. Klein, l. c. p. 115.

‡ Cf. F. Klein, l. c. p. 49.

We have, then, only to seek, for each of the above enumerated groups, all the cases where the expression (4) represents a sextic.

Excluding the cases where f has equal roots, we easily find the following result:

Any binary sextic f with linear transformations into itself (except of course the transformations $x_1 = \pm x_1'$, $x_2 = \pm x_2'$), and whose roots are all different from each other, is reducible, by a linear substitution, *to one of the following canonical forms:*

$$\left.\begin{array}{rl}
\text{I.} & f = z_1^6 + \alpha z_1^4 z_2^2 + \beta z_1^2 z_2^4 + z_2^6 \\
 & \qquad (\textit{Cyclic group } n = 2) \\
\text{II.} & f = z_1 (z_1^5 + z_2^5) \\
 & \qquad (\textit{Cyclic group } n = 5) \\
\text{III.} & f = z_1 z_2 (z_1^4 + \alpha z_1^2 z_2^2 + z_2^4) \\
 & \qquad (\textit{Diedron group } n = 2) \\
\text{IV.} & f = z_1^6 + \alpha z_1^3 z_2^3 + z_2^6 \\
 & \qquad (\textit{Diedron group } n = 3) \\
\text{V.} & f = z_1^6 + z_2^6 \\
 & \qquad (\textit{Diedron group } n = 6) \\
\text{VI.} & f = z_1 z_2 (z_1^4 + z_2^4) \\
 & \qquad (\textit{Oktaedron group})
\end{array}\right\} \qquad (A)$$

The inequalities for the parameters α, β which must be added in order to discriminate between the different cases, are given in the following section.

SECTION II.

The Relations between the Rational Invariants.

§3.

We now purpose to find the necessary and sufficient criteria that a given binary sextic be reducible by linear transformation to one of the above enumerated canonical forms.

These special sextics have been examined by Clebsch in his "Theorie der binaeren algebraischen Formen," and more recently by Maisano in a paper "Sulla sestica binaria" (Atti della R. Accademia dei Lincei, 1883–84).

For the first two cases the criteria in question are completely given by Clebsch in the form of relations between the invariants of the sextic. Using Clebsch's notation,[*] they are

$$\begin{aligned} &\text{I. } f = z_1^6 + \alpha z_1^4 z_2^2 + \beta z_1^2 z_2^4 + z_2^6, \\ &\quad R = 0,\ A_{ll}A_{mm} - A_{lm} \gtrless 0.\dagger \end{aligned} \tag{5}$$

$$\begin{aligned} &\text{II. } f = z_1(z_1^5 + z_2^5), \\ &\quad A = 0,\ B = 0,\ C = 0,\ D \gtrless 0.\ddagger \end{aligned} \tag{6}$$

The criteria in the other cases are given by Maisano as the vanishing of certain *covariants*. But in these cases also they can be expressed by relations between *invariants*, as we shall show.

The subject is closely connected with the determination of the conditions that a sextic f be transformable into another f' by linear substitution. In general, the equality of the corresponding rational absolute invariants of f and f' is necessary *and sufficient*, as Clebsch has proved[§] with the aid of the typical representation ("typische Darstellung") of the binary sextic by means of quadratic covariants. But whether the same condition is sufficient in those exceptional cases where the typical representation is impossible, has not yet been investigated, so far as I know. And as the canonical forms III to VI of our table (A) are the most important of these exceptional cases, I shall answer this question in what follows. We denote with Clebsch:

$$\left.\begin{aligned} &f = a_x^6 = a_0x_1^6 + 6a_1x_1^5x_2 + 15a_2x_1^4x_2^2 + 20a_3x_1^3x_2^3 + 15a_4x_1^2x_2^4 + 6a_5x_1x_2^5 + a_6x_2^6, \\ &i = (ab)^4a_x^2b_x^2 = \alpha_0x_1^4 + 4\alpha_1x_1^3x_2 + 6\alpha_2x_1^2x_2^2 + 4\alpha_3x_1x_2^3 + \alpha_4x_2^4, \\ &l = (ai)^4a_x^2,\ m = (il)^2i_x^2,\ n = (im)^2i_x^2, \\ &A = (ab)^6,\ B = (ii')^4,\ C = (ii')^2(i'i'')^2(i''i)^2, \\ &D = (nl)^2,\ R = -(lm)(mn)(nl). \end{aligned}\right\} \tag{7}$$

Moreover,

$$\left.\begin{aligned} A_{ll} &= (ll')^2 = 2C + \frac{1}{3}AB, \\ A_{lm} &= (lm)^2 = \frac{2}{3}(B^2 + AC), \\ A_{ln} &= (ln)^2 = D = (mm') = A_{mm}, \\ A_{mn} &= (mn)^2 = \frac{1}{2}BA_{lm} + \frac{1}{3}CA_{ll}, \\ A_{nn} &= (nn')^2 = \frac{1}{2}BD + \frac{1}{3}CA_{lm}. \end{aligned}\right\} \tag{8}$$

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[*] Cf. the following page.

[†] Cf. Clebsch, l. c. p. 455.

[‡] Cf. Clebsch, l. c. p. 457.

[§] L. c. p. 421.

The typical representation is then, and only then, impossible when the three quadratic covariants l, m, n have a common factor. According to the investigations of Clebsch and Maisano, this can happen in two different ways; either l vanishes identically or, l being different from zero, m differs from l only by a constant factor.*

§4.

Let us first examine the case

$$l \equiv 0. \tag{9}$$

We then have from the formulæ (8),

$$A_{ll} = 0,\ A_{lm} = 0,\ D = 0. \tag{10}$$

Conversely, from these three equations follows $l \equiv 0$. For A_{ll} being $= 0$, we may put

$$l = l_0 x_1^2.$$

If l_0 were different from zero, we should infer from

$$A_{lm} = l_0 m_2 = 0,$$

that

$$m_2 = 0,$$

and from

$$D = -2m_1^2 = 0, \text{ that } m_1 = 0.$$

Therefore we should have

$$m = l_0 \alpha_2 x_1^2;$$

that is to say, m would differ from l only by a constant factor. Now, Clebsch has proved the two lemmas,† "If m vanishes identically, l must vanish also," and "If $m =$ const. l, and $l \gtrless 0$, l cannot be a complete square." Thence follows that l_0 must be $= 0$; we have therefore the lemma:

The covariant condition $l \equiv 0$ is equivalent to the invariant conditions

$$A_{ll} = 0,\ A_{lm} = 0,\ D = 0.$$

Now, if for a binary sextic f the covariant l vanishes identically, the following cases are to be distinguished:‡

1). If at the same time $A = 0$,

the sextic f has a quadruple, a quintuple, or a sextuple factor, and the three other invariants vanish also,

$$B = 0,\ C = 0,\ D = 0.$$

2). On the contrary, if $A \gtrless 0$,

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[*] Clebsch, l. c. p. 437, and Maisano, l. c. p. 32.

[†] L. c. p. 445 and p. 441.

[‡] Cf. Maisano, l. c. p. 32.

the sextic f is, by a linear substitution of the determinant 1, reducible to one of the two forms

$$a\,(z_1^6 + z_2^6),\quad a \gtrless 0,$$

or

$$6az_1z_2\,(z_1^4 + z_2^4),\quad a \gtrless 0.$$

In the former case we have besides $B \gtrless 0$, in the latter, $B = 0$.

Herewith are the criteria for the two cases V and VI of the table (A) reduced to relations between invariants, but they admit of further simplification:

a) $$f = a\,(z_1^6 + z_2^6),\quad a \gtrless 0;$$

then we may replace the equation $A_{lm} = 0$ of the system (10) by $6B - A^2 = 0$, and the inequality $B \gtrless 0$ follows from $A \gtrless 0$, so that the criteria for this case are

$$6B - A^2 = 0,\quad 2C + \frac{1}{3}AB = 0,\quad D = 0, \tag{11}$$
$$A \gtrless 0.$$

b) $$f = 6az_1z_2\,(z_1^4 + z_2^4),\quad a \gtrless 0;$$

here the criteria are
$$B = 0,\quad C = 0,\quad D = 0, \tag{12}$$
$$A \gtrless 0.$$

§5.

We pass to the case where $l \gtrless 0$ and m differs from l only by a constant factor; that is,
$$m = kl,\quad \text{or},\quad (lm)\,l_x m_x \equiv 0; \tag{13}$$
then, according to the two lemmas of Clebsch's cited in the foregoing paragraph, k is certainly $\gtrless 0$ and the two factors of l are different from each other, therefore $A_{ll} \gtrless 0$; besides, $n = k^2 l$; thence follows

$$DA_{ll} = A_{lm}^2,\quad A_{mn}A_{ll} = A_{lm}D.$$

Conversely, from these two equations and the inequality $A_{ll} \gtrless 0$ follows the relation (13).

For, A_{ll} being $\gtrless 0$, we may write

$$l = 2l_1x_1x_2,\quad \text{and}\quad l_1 \gtrless 0.$$

Thence follow

$$\begin{aligned}
m &= -\,2l_1\,(\alpha_1x_1^2 + 2\alpha_2x_1x_2 + \alpha_3x_2^2),\\
n &= -\,2l_1\big((\alpha_0\alpha_3 - \alpha_1\alpha_2)\,x_1^2 + 4\,(\alpha_1\alpha_3 - \alpha_2^2)\,x_1x_2 + (\alpha_1\alpha_4 - \alpha_2\alpha_3)\,x_2^2\big),\\
A_{ll} &= -\,2l_1^2,\quad A_{lm} = 4l_1^2\alpha_2,\quad A_{ln} = 8l_1^2\,(\alpha_1\alpha_3 - \alpha_2^2).
\end{aligned}$$

The equation $DA_{ll} = A_{lm}^2$ then becomes

$$\alpha_1\alpha_3 = 0.$$

We may assume $\alpha_1 = 0$, then we have

$$A_{mn} = 4l_1^2(4\alpha_2^3 + \alpha_0\alpha_3^2), \ A_{ln} = -8l_1^2\alpha_2^2,$$

therefore the equation $\qquad A_{mn}A_{ll} = A_{lm}D$

becomes $\qquad \alpha_0\alpha_3 = 0,$

and the three covariants l, m, n have the common factor x_2, l being at the same time $\gtrless 0$. Now, Maisano has proved,* "If l, m, n have a common factor, they have also the other factor common"; we have then the lemma:

The covariant conditions

$$(lm)\,l_x m_x \equiv 0, \ l \gtrless 0 \tag{13}$$

are equivalent to the invariant conditions

$$DA_{ll} = A_{lm}^2, \ A_{mn}A_{ll} = A_{lm}D, \tag{14}$$
$$A_{ll} \gtrless 0.$$

Now, from the relations (13) follows that the sextic f is reducible by a linear substitution of the determinant 1 to one of the three forms †

$$20a_3z_1^3z_2^3 + a_5z_1z_2^5 + a_6z_2^6, \ a_3 \gtrless 0,$$
$$a(z_1^6 + \alpha z_1^3z_2^3 + z_2^6), \ a\alpha(\alpha^2 + 50) \gtrless 0, ‡$$
$$6az_1z_2(z_1^4 + \alpha z_1^2z_2^2 + z_2^4), \ a\alpha(9\alpha^2 - 100) \gtrless 0. §$$

In the first case we easily find $\quad C^2 - \frac{1}{6}B^3 = 0, \ 2AB - 15C = 0;$

in the second case, $\quad C^2 - \frac{1}{6}B^3 = 0, \ 2AB - 15C \gtrless 0;$ $\qquad$ (15)

in the third case, $\quad C^2 - \frac{1}{6}B^3 \gtrless 0.$

Therefore the relations (14), together with $C^2 - \frac{1}{6}B^3 = 0$, $2AB - 15C = 0$, are the necessary and sufficient conditions that a sextic be reducible by linear substitution to the form

$$20a_3z_1^3z_2^3 + a_5z_1z_2^5 + a_6z_2^6,$$

and similarly for the two other cases. But these conditions can be further simplified. An easy calculation leads to the following results:

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* L. c. p. 34.

† Cf. Maisano, l. c. p. 34.

‡ If $\alpha^2 + 50$ were $= 0$, the sextic would be reducible to $6a'z_1'z_2'(z_1'^4 + z_2'^4)$.

§ If $9\alpha^2 - 100$ were $= 0$, the sextic would be reducible to $a'(z_1'^6 + z_2'^6)$.

a). For
$$f = 20a_3z_1^3z_2^3 + a_5z_1z_2^5 + a_6z_2^6, \qquad (16)$$
$$a_3 \gtrless 0,$$
the necessary and sufficient conditions are*
$$3.5^2.B = 2^3.A^2, \; 3^2.5^3.C = 2^4.A^3, \; 3^2.5^5D = 2^7.A^5, \qquad (17)$$
$$A \gtrless 0.$$
Moreover,
$$a_3 = \sqrt{-\frac{A}{2^2.5}},$$
a_5 and a_6 are completely independent of the invariants.

b). For
$$f = a(z_1^6 + \alpha z_1^3z_2^3 + z_2^6), \qquad (18)$$
$$a\alpha(\alpha^2 + 50) \gtrless 0,$$
the necessary and sufficient conditions are
$$C^2 - \frac{1}{6}B^3 = 0, \; 9D - 2B(6C + AB) = 0, \qquad (19)$$
$$D \gtrless 0, \; 2AB - 15C \gtrless 0.$$
Moreover,
$$\left.\begin{aligned} a &= \frac{1}{3}\sqrt{\frac{2AB - 15C}{B}} \\ \alpha &= 10\sqrt{\frac{6C + AB}{15C - 2AB}} \end{aligned}\right\} \qquad (20)$$
as from the conditions (19) follow:
$$A_{ll} \gtrless 0, \; B \gtrless 0,$$
the expressions of a and α can never become infinite or indeterminate.

Finally, we have for the covariant l the expression
$$l = \sqrt{-2A_{ll}}\,z_1z_2. \qquad (21)$$

c). For
$$f = 6az_1z_2(z_1^4 + \alpha z_1^2z_2^2 + z_2^4), \qquad (22)$$
$$a\alpha(9\alpha^2 - 100) \gtrless 0,$$
the necessary and sufficient conditions are
$$\left.\begin{aligned} 3AB^2 - 6BC + 4A^2C - 18D &= 0 \\ 4B^3 + 5ABC + 6C^2 - 3AD &= 0 \\ D \gtrless 0, \; C^2 - \frac{1}{6}B^3 &\gtrless 0 \end{aligned}\right\} \qquad (23)$$
In this case, the proof being more complicated, we shall indicate it in a few words. The two equations (23) can be written in the form
$$6D = 2AA_{lm} - BA_{ll},$$
$$AD = 2BA_{lm} + CA_{ll}.$$

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* These conditions are given by Maisano, l. c. p. 35.

Thence we deduce

$$DA_{ll} = A_{lm}^2, \quad -D(6B - A^2) = \frac{3}{2} A_{ll}A_{lm}, \quad A_{mn}A_{lm} = D^2.$$

D being supposed $\gtrless 0$, we have also $A_{mn} \gtrless 0$, $A_{lm} \gtrless 0$, and from $DA_{ll} = A_{lm}^2$ also $A_{ll} \gtrless 0$. We may then divide the two equations

$$\begin{aligned} A_{mn}A_{lm} &= D^2, \\ A_{lm}^2 &= DA_{ll} \end{aligned}$$

by each other; we thus obtain

$$A_{mn}A_{ll} = DA_{lm}.$$

Therefore the three conditions (14) can be deduced from the two equations (23) and the inequality $D \gtrless 0$. But the conditions (14), together with the inequality $C^2 - \frac{1}{6}B^3 \gtrless 0$, are sufficient, as we have seen above.

As to the parameters a and α, we have

$$\left.\begin{aligned} a &= \frac{1}{2}\sqrt{\frac{C^2 - \frac{1}{6}B^3}{6D}} \\ \alpha &= 10\sqrt{\frac{B^2 + AC}{2A^2B - 3AC - 15B^2}} \end{aligned}\right\} \tag{24}$$

For the covariant l we find

$$l = \sqrt{-2A_{ll}z_1z_2}. \tag{25}$$

§6.

We can now answer the question proposed in §3. Suppose f or f' to be one of the above enumerated exceptional sextics for which the typical representation is impossible; and suppose it be possible to find a quantity r different from zero, such that between the invariants of the two sextics the relations exist:

$$A' = r^6A, \quad B' = r^{12}B, \quad C' = r^{18}C, \quad D' = r^{30}D, \tag{26}$$

we must then examine whether these conditions are sufficient for the possibility of transformation of f into f' by linear substitution.

We choose as an example the case where between the invariants of f the relations (19) exist:

$$\begin{gathered} C^2 - \frac{1}{6}B^3 = 0, \quad 9D - 2B(6C + AB) = 0, \\ D \gtrless 0, \quad 2AB - 15C \gtrless 0. \end{gathered}$$

We can then reduce f to the form

$$z_1^6 + \alpha z_1^3 z_2^3 + z_2^6,$$

where

$$\alpha = 10\sqrt{\frac{6C + AB}{15C - 2AB}}.$$

From (26) we have also between the invariants of f' the relations

$$C'^2 - \frac{1}{6} B'^3 = 0, \; gD' - 2B'(6C' + A'B') = 0,$$

$$D' \gtrless 0, \; 2A'B' - 15C' \gtrless 0,$$

and we can therefore reduce f' to the form

$$z_1'^6 + \alpha' z_1'^3 z_2'^3 + z_2'^6,$$

where

$$\alpha' = 10\sqrt{\frac{6C' + A'B'}{15C' - 2A'B'}}.$$

But from (26) we have $\alpha' = \pm \alpha$.

We may assume $\alpha' = + \alpha$; for, if α' were $= - \alpha$, we should return to the former case by applying to the variables $z_1' z_2'$ the substitution $z_1' = z_1''$, $z_2' = - z_2''$. Thence we immediately infer that f can be transformed into f' by linear substitution.

Exactly the same considerations are applicable to the cases where f is reducible to one of the canonical forms

$$z_1 z_2 (z_1^4 + \alpha z_1^2 z_2^2 + z_2^4),$$

$$z_1^6 + z_2^6,$$

$$z_1 z_2 (z_1^4 + z_2^4).$$

But the proof evidently fails in the cases where f has a quadruple, a quintuple, or a sextuple linear factor; for it is impossible to discriminate between these three cases by invariant conditions, as in each case we have

$$A = 0, \; B = 0, \; C = 0, \; D = 0.$$

Finally, in the case of a triple factor,

$$f = 20 a_3 z_1^3 z_2^3 + 6 a_1 z_1 z_2^5 + a_6 z_2^6,$$

the equality of the absolute invariants is likewise insufficient. For, as the absolute invariants are, in this case, completely independent of a_5 and a_6, the three cases of—

1). One triple and three simple factors,
2). One triple, one double, and one simple factor,
3). Two triple factors,

give the same values of the absolute invariants of f.

The exceptional cases being thus completely discussed, we may express the theorem concerning the transformation of two sextics into each other in the following form:

Two binary sextics which have no triple, quadruple, quintuple, or sextuple factors, can be transformed into each other by linear substitution, if their corresponding rational absolute invariants are equal; but the equality of the absolute invariants is not sufficient if one of the two sextics has such a multiple factor.

§7.

We now revert to the binary sextics with linear transformations into themselves; the criteria that a sextic be reducible to one of the canonical forms (A) being established in §§3, 4, 5, there still remains to determine *the linear substitutions by means of which this reduction is to be performed.*

In each of the *four first cases* of the table (A) there exists a quadratic covariant which, by introducing the canonical variables z_1, z_2, takes the form

$$\text{Const. } z_1 z_2 .$$

We then obtain the substitution in question by putting z_1 and z_2 equal to the two linear factors of that quadratic covariant, multiplied by suitable constant factors. For the first case the quadratic covariant is $(lm)\, l_x m_x$;[*] for the second, m;[†] for the third and fourth, l.[‡]

In the two last cases all the quadratic covariants vanish identically; a different method must therefore be applied, which is founded on the fact that the determination of the canonical variables z_1, z_2 is identical with the solution of the form-problem ("Formenproblem"[§]) belonging to the group of linear substitutions in question.

In the case of the *diedron group* $n = 6$ (case V of our table), the absolutely invariant groundforms are[‖]

$$\Phi_1 = (z_1 z_2)^2, \quad \Phi_2 = \left(\frac{z_1^6 + z_2^6}{2}\right)^2, \quad \Phi_3 = \frac{z_1 z_2 (z_1^{12} - z_2^{12})}{4}. \tag{27}$$

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[*] Cf. Clebsch, l. c. p. 457 and p. 199.

[†] *Ibid.* p. 467.

[‡] Cf. the equations (21) and (25).

[§] By the "Formenproblem" belonging to a finite group G of homogeneous linear substitutions of the variables z_1, z_2, Klein understands the following problem: The values of all the invariant groundforms of the group G being given, to determine the corresponding values of the variables z_1, z_2.

[‖] Cf. Klein, Vorlesungen über das Ikosaeder, p. 63.

They are found to be covariants of the sextic f, viz.

$$\Phi_1 = \frac{i}{A}, \quad \Phi_2 = \frac{f^2}{2A}, \quad \Phi_3 = \frac{fT}{\sqrt{2Ai}}, \tag{28}$$

where $$T = (aH)\, a_x^5 H_x^7.$$

To find the canonical variables z_1, z_2, we have therefore to solve the problem: The values of the invariant groundforms of the diedron group $n = 6$ being given, to determine the corresponding values of the variables z_1, z_2, and this is exactly the form-problem belonging to the group in question.

The solution follows immediately from the equations

$$\frac{z_1^6 + z_2^6}{2} = \frac{f}{\sqrt{2A}}.$$
$$\frac{z_1^6 - z_2^6}{2} = \sqrt{\frac{A^2 f^2 - 2i^3}{2A^3}}.$$

For the case of the *oktaedron group* (case VI of our table), the invariant groundforms are*

$$\left.\begin{aligned} \Phi_1 &= [z_1 z_2 (z_1^4 + z_2^4)]^2, \\ \Phi_2 &= z_1^8 - 14 z_1^4 z_2^4 + z_2^8, \\ \Phi_3 &= [z_1^{12} + 33 z_1^8 z_2^4 - 33 z_1^4 z_2^8 - z_2^{12}][z_1 z_2 (z_1^4 + z_2^4)]; \end{aligned}\right\} \tag{29}$$

they are covariants of the sextic f, viz.

$$\Phi_1 = -\frac{f^2}{3A}, \quad \Phi_2 = \frac{6H}{A}, \quad \Phi_3 = \frac{12fT}{A^2}. \tag{30}$$

The determination of z_1 and z_2 can be performed with the aid of the resolvent of the third degree,

$$16A\phi^3 + 6H\phi - \frac{f^2}{3} = 0,$$

the three roots of which are

$$\phi_1 = (z_1 z_2)^2, \quad \phi_2 = \frac{(z_1^2 - i z_2^2)^2}{4i}, \quad \phi_3 = -\frac{(z_1^2 + i z_2^2)^2}{4i},$$

whence z_1 and z_2 can immediately be calculated.

The same method is applicable to the determination of the canonical variables in the other cases of the table (A), but the formulæ are rather complicated.

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* Cf. Klein, l. c. p. 63.

SECTION III.

The Relations between the ϑ-Moduli.

§8.

The conditions between the *algebraic* invariants of our special sextics being established in the foregoing section, we now proceed to determine the corresponding conditions between the *transcendental absolute invariants;* that is to say, between the ϑ-moduli $\tau_{\alpha\beta}$.

Using homogeneous variables, we put, with Klein,[*]

$$\left.\begin{aligned} u_1 &= \int^{z_1} \frac{(z_1 dz_2 - z_2 dz_1)}{\sqrt{f(z_1 z_2)}} \\ u_2 &= \int^{z_2} \frac{(z_1 dz_2 - z_2 dz_1)}{\sqrt{f(z_1 z_2)}} \end{aligned}\right\} \tag{31}$$

the lower limit of the integrals being any fixed point in the Riemann's surface belonging to the irrationality $\frac{\sqrt{f(z_1, z_2)}}{z_2^3}$. Let

$$\begin{array}{c|cccc|} u_1 & \omega_{11}, & \omega_{12}, & \omega_{13}, & \omega_{14} \\ u_2 & \omega_{21}, & \omega_{22}, & \omega_{23}, & \omega_{24} \end{array} \tag{32}$$

be a system of simultaneous normal periods[†] for the two integrals, and let K_i be the path of integration for the periods ω_{1i}, ω_{2i}. Putting, then, for shortness

$$p_{ik} = \omega_{1i}\omega_{2k} - a_{1k}\omega_{2i},$$

the ϑ-moduli $\tau_{\alpha\beta}$ of Weierstrass are defined by the equations

$$\tau_{11} = \frac{p_{32}}{p_{12}}, \quad \tau_{12} = \frac{p_{42}}{p_{12}} = \frac{p_{13}}{p_{12}} = \tau_{21}, \quad \tau_{22} = \frac{p_{14}}{p_{12}}. \tag{33}$$

In order to determine the relations which must exist between these magnitudes $\tau_{\alpha\beta}$, if the sextic $f(z_1 z_2)$ be one of our canonical forms (A), we first apply to the variables any linear substitution

$$\left.\begin{aligned} z_1 &= p z_1' + q z_2' \\ z_2 &= p' z_1' + q' z_2' \\ \sqrt{f(z_1, z_2)} &= \sqrt{f'(z_1', z_2')} \end{aligned}\right\} \tag{34}$$

Denoting, then, by u_1', u_2', the integrals

$$\left.\begin{aligned} u_1' &= \int^{z_1'} \frac{(z_1' dz_2' - z_2' dz_1')}{\sqrt{f'(z_1', z_2')}} \\ u_2' &= \int^{z_2'} \frac{(z_1' dz_2' - z_2' dz_1')}{\sqrt{f'(z_1', z_2')}} \end{aligned}\right\} \tag{35}$$

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[*] F. Klein, Hyperelliptische Sigmafunctionen, Math. Annalen, Bd. 27.

[†] "Normale Periodicitätsmoduln," according to Clebsch and Gordan, or "Périodes normales," according to Briot.

and by ω'_{1i} and ω'_{2i} those periods of u'_1 and u'_2 which belong to the path of integration K'_i, corresponding to the path K_i, we have for the magnitudes $\tau'_{\alpha\beta}$ the fundamental equations*

$$\tau'_{\alpha\beta} = \tau_{\alpha\beta}\,; \tag{36}$$

that is to say, the $\tau_{\alpha\beta}$ are transcendental absolute invariants of the binary sextic f.

The equation (36) holds good whatever the sextic f and the substitution (34) may be. Let us now suppose f to be one of the six canonical forms (A) and (34) one of the substitutions leaving the function f unchanged, so that

$$f'(z'_1, z'_2) = cf(z'_1, z'_2). \tag{37}$$

Assuming for simplicity $c = 1$, the magnitudes

$$\omega'_{11},\ \omega'_{12},\ \omega'_{13},\ \omega'_{14}$$

are a system of normal periods *of the same integral* u_1, differing from

$$\omega_{11},\ \omega_{12},\ \omega_{13},\ \omega_{14}$$

only by the path of integration. Therefore the periods ω'_{1i} can be expressed by the ω_{1i} by means of a linear substitution,

$$\omega'_{1i} = m_{i1}\omega_{11} + m_{i2}\omega_{12} + m_{i3}\omega_{13} + m_{i4}\omega_{14}, \tag{ε}$$

and likewise $\qquad \omega'_{2i} = m_{i1}\omega_{21} + m_{i2}\omega_{22} + m_{i3}\omega_{23} + m_{i4}\omega_{24},$

the integers m_{ik} satisfying the known bilinear relations.

Thence follows a system of formulæ expressing $\tau'_{\alpha\beta}$ by $\tau_{11}, \tau_{12}, \tau_{22}$, which, for shortness, we write

$$\tau'_{\alpha\beta} = \phi_{\alpha\beta}(\tau_{11}, \tau_{12}, \tau_{22}). \tag{38}$$

The $\tau_{\alpha\beta}$ being homogeneous functions of order zero of the $\omega_{\alpha\beta}$, we obtain the same equations (38) in the case where the constant factor c in (37) is different from unity. Combining the two equations (36) and (38) we have

$$\tau_{\alpha\beta} = \phi_{\alpha\beta}(\tau_{11}, \tau_{12}, \tau_{22})\,; \tag{39}$$

that is, three relations between the moduli $\tau_{\alpha\beta}$, and these are exactly the relations we had proposed to find in the beginning of this paragraph. Interpreting the relations (39) in a little different way, we have the following proposition:

If the sextic f remains unchanged for a group of linear substitutions of the variables z_1, z_2, any system of ϑ-moduli $\tau_{11}, \tau_{12}, \tau_{22}$ belonging to $\sqrt{f}$ remains likewise unchanged for a corresponding group of linear transformations of the periods.

And we may add as a corollary: Of these linear transformations of the

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* Cf. Klein, Hyperelliptische Sigmafunctionen, Math. Ann. Bd. 27, p. 437.

periods, only those can be congruent to identity modulo 2* which correspond to the substitutions
$$z_1 = \pm z_1', \; z_2 = \pm z_2',$$
as we easily infer from the conclusions of §1.

§9.

Conversely, if the system of ϑ-moduli $\tau_{\alpha\beta}$ remains unchanged for a linear transformation s of the periods, not congruent to identity modulo 2, the sextic f remains unchanged for a corresponding linear substitution S of the variables z_1, z_2, different from
$$z_1 = \pm z_1', \; z_2 = \pm z_2'.$$
To prove this we transform the sextic f into the canonical form of Richelot belonging to the system of ϑ-moduli $\tau_{\alpha\beta}$. If, to fix the ideas, we suppose the path of integration K_1 to include the two branching points ("Verzweigungspuncte") α_1 and α_2, K_2 the points α_3 and α_4, K_3 the points α_6 and α_1, K_4 the points α_4 and α_5, Richelot's moduli $\varkappa^2$, λ^2, μ^2 are given by the formulæ
$$\varkappa^2 = \frac{(\alpha_3 - \alpha_4)(\alpha_2 - \alpha_5)}{(\alpha_3 - \alpha_5)(\alpha_2 - \alpha_4)} = \frac{\vartheta_{23}^2 \vartheta_4^2}{\vartheta_{01}^2 \vartheta_5^2},$$
$$\lambda^2 = \frac{(\alpha_3 - \alpha_4)(\alpha_1 - \alpha_5)}{(\alpha_3 - \alpha_5)(\alpha_1 - \alpha_4)} = \frac{\vartheta_{03}^2 \vartheta_{23}^2}{\vartheta_{12}^2 \vartheta_{01}^2},$$
$$\mu^2 = \frac{(\alpha_3 - \alpha_4)(\alpha_6 - \alpha^5)}{(\alpha_3 - \alpha_5)(\alpha_6 - \alpha_4)} = \frac{\vartheta_{03}^2 \vartheta_4^2}{\vartheta_{12}^2 \vartheta_5^2},$$
ϑ_α denoting
$$\vartheta_\alpha = \vartheta(0, 0; \tau_{11}, \tau_{12}, \tau_{22})_\alpha.$$
The sextic f is then transformed into
$$F = \text{Const.}\; y_1 y_2 (y_2 - y_1)(y_2 - \varkappa^2 y_1)(y_2 - \lambda^2 y_1)(y_2 - \mu^2 y_1)$$
by means of the substitution
$$\begin{aligned} y_1 &= (\alpha_3 - \alpha_5)(z_1 - \alpha_4 z_2), \\ y_2 &= (\alpha_3 - \alpha_4)(z_1 - \alpha_5 z_2). \end{aligned} \tag{T}$$
Suppose, now, τ_{11}', τ_{12}', τ_{22}' to be any other system of ϑ-moduli belonging to the same sextic f and derived from τ_{11}, τ_{12}, τ_{22} by means of a linear transformation s of the periods, then the corresponding moduli of Richelot, $\varkappa'^2$, λ'^2, μ'^2, can be derived from $\varkappa^2$, λ^2, μ^2 by applying a certain permutation
$$\sigma = \begin{pmatrix} \alpha_1, & \alpha_2, & \alpha_3, & \alpha_4, & \alpha_5, & \alpha_6 \\ \alpha_1', & \alpha_2', & \alpha_3', & \alpha_4', & \alpha_5', & \alpha_6' \end{pmatrix}$$

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*A linear transformation of the periods is said to be "congruent to identity modulo k," if all the coefficients are divisible by k except those in the principal diagonal, the latter being congruent to 1 (modulo k).

to the roots α'_i, and the sextic f passes into the form

$$F' = \text{Const.}\ y'_1 y'_2 (y'_2 - y'_1)(y'_2 - \kappa'^2 y'_1)(y'_2 - \lambda'^2 y'_1)(y'_2 - \mu'^2 y'_1)$$

by means of the substitution

$$\begin{aligned} y'_1 &= (\alpha'_3 - \alpha'_5)(z_1 - \alpha'_4 z_2), \\ y'_2 &= (\alpha'_3 - \alpha'_4)(z_1 - \alpha'_5 z_2). \end{aligned} \qquad (\mathrm{T}')$$

If, now, we suppose the transformation s to be one of those which leave the $\tau_{\alpha\beta}$ unchanged, so that

$$\tau'_{\alpha\beta} = \tau_{\alpha\beta},$$

we have

$$\vartheta(0, 0;\ \tau'_{11}, \tau'_{12}, \tau'_{22})_\alpha = \vartheta(0, 0;\ \tau_{11}, \tau_{12}, \tau_{22})_\alpha,$$

and therefore

$$\kappa'^2 = \kappa^2,\ \lambda'^2 = \lambda^2,\ \mu'^2 = \mu^2.$$

Thence we infer that the sextic f is only changed by a constant factor if we apply to the variables z_1, z_2 the combined substitution $T'T^{-1}$. Moreover, if the substitution s is not congruent to identity modulo 2, the permutation σ is different from identity, and therefore the substitution $T'T^{-1}$, considered as non-homogeneous, is different from identity too. Thence the substitution $T'T^{-1}$, considered as homogeneous and reduced to the determinant 1, is different from $z_1 = \pm z'_1$, $z_2 = \pm z'_2$, as was to be proved.

§10.

We now proceed to the application of the method above explained to each of our canonical forms (A).

I. Case: Using non-homogeneous variables

$$z = \frac{z_1}{z_2},$$

we write

$$f(z) = z^6 + \alpha z^4 + \beta z^2 + 1.$$

The roots of $f(z)$ being in pairs equal and of opposite signs, we choose their order so that

$$\alpha_2 = -\alpha_1,\ \alpha_4 = -\alpha_3,\ \alpha_6 = -\alpha_5.$$

The periods may be calculated in the following manner:

Starting from the point α_1, we join the branching points in their natural order by any line L nowhere cutting itself; and we call that side of the line positive which is on the left when we pass along the line in the direction from α_1 through α_2 . . . . to α_6. We choose as connections* ("Verzweigungsschnitte") of our Riemann's surface T the portions $\alpha_1\alpha_2$, $\alpha_3\alpha_4$, $\alpha_5\alpha_6$ of the line L, and we define as the upper leaf the one for which

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* The expression "connection" has been introduced by Cole.

$$\sqrt{f(0)} = +1.$$

Denoting, then, by $F_{\lambda\mu}$ either of the integrals

$$\int_{a_\lambda}^{a_\mu} \frac{z\,dz}{\sqrt{f(z)}}, \quad \int_{a_\lambda}^{a_\mu} \frac{dz}{\sqrt{f(z)}},$$

taken along the positive side of the line L in the upper leaf of the surface, ω_i being accordingly ω_{1i} or ω_{2i}, we may write

$$\omega_1 = 2F_{12}, \quad \omega_2 = 2F_{34},$$
$$\omega_3 = 2F_{16}, \quad \omega_4 = 2F_{45}.$$

The substitution $\qquad z' = -z$

leaves $f(z)$ unchanged: $\qquad f(z') = f(z),$

and the line L is changed into the line L' obtained by turning the whole plane through an angle of 180° around the point $z = 0$; by this operation the branching points are interchanged according to the permutation

$$\sigma = (\alpha_1\alpha_2)(\alpha_3\alpha_4)(\alpha_5\alpha_6),$$

or $\qquad \alpha'_1 = \alpha_2, \; \alpha'_2 = \alpha_1$, etc.

We now construct a second Riemann's surface T' with the connections $\alpha'_1\alpha'_2$, $\alpha'_3\alpha'_4$, $\alpha'_5\alpha'_6$ along L', the upper leaf being again defined by

$$\sqrt{f(0)} = +1,$$

and we complete the correspondence of T and T' determined by the substitution $z' = -z$ by requiring that the point $z = 0$, $\sqrt{f(z)} = +1$ of T corresponds to the point $z' = 0$, $\sqrt{f(z')} = +1$ of T'.

Denoting by $\psi(z)$ the value of $\sqrt{f(z)}$ in the point z of the upper leaf of T, and by $\psi'(z)$ the value of $\sqrt{f(z)}$ in the same point z of the upper leaf of T', we easily find the plane of the variable z to be divisible into four annular portions (see Fig. 1), so that $\dfrac{\psi'(z)}{\psi(z)} = +1$ in the shaded and $= -1$ in the blank portions, the full lines in the figure representing parts of the line L; the dotted, parts of L'.

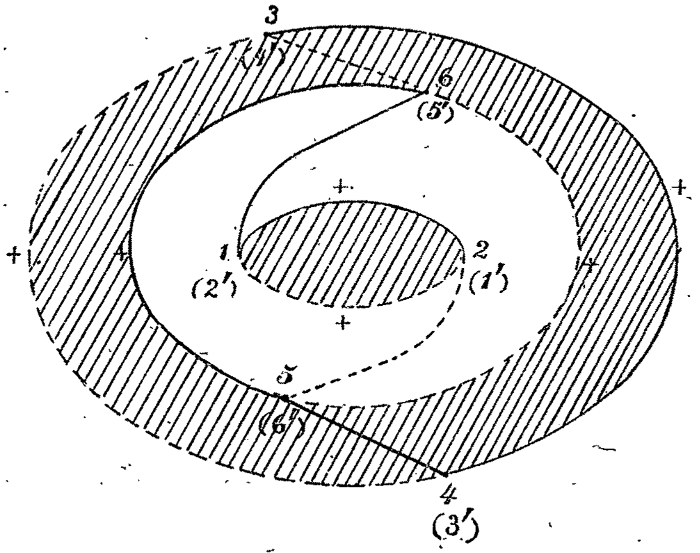

Fig. 1.

We may now easily determine the transformation s. For instance,

$$\omega_1' = 2I'_{1'2'},$$

where $I'_{1'2'}$ extends from $1'$ to $2'$ along the positive side of L' in the upper leaf of T'. We see from the figure that along this path

$$\psi'(z) = -\psi(z);$$

we may therefore calculate $I'_{1'2'}$ also in the surface T along L' *in the lower leaf*, or by deforming gradually the path of integration, along the negative side of L in the lower leaf, that is to say, along the positive side of L in the upper leaf from α_2 to α_1. Therefore

$$\omega_1' = 2I'_{1'2'} = + 2I_{21} = - 2I_{12} = -\omega_1.$$

In like manner we find

$$\begin{aligned} \omega_2' &= 2I'_{3'4'} = + 2I_{34} = \omega_2, \\ \omega_3' &= 2I'_{1'6'} = - 2F_{25} = - 2F_{16} + 2I_{34} = \omega_2 - \omega_3, \\ \omega_4' &= 2I'_{4'5'} = + 2F_{30} = \quad 2F_{45} - 2I_{12} = \omega_4 - \omega_1, \end{aligned}$$

which is, indeed, a "canonical substitution"* of the determinant 1.

The corresponding formulæ for the transformation of the ϑ-moduli are

$$\tau_{11}' = \tau_{11}, \quad \tau_{12}' = 1 - \tau_{12}, \quad \tau_{22}' = \tau_{22},$$

whence we have, putting $\tau'_{\alpha\beta} = \tau_{\alpha\beta}$,

$$\tau_{12} = \frac{1}{2}$$

as the relation between the $\tau_{\alpha\beta}$ which we seek.

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* Cf. Clebsch and Gordan, Abel'sche Functionen, p. 300.

The result agrees exactly with the following theorem of Weierstrass* and Picard† concerning the reduction of hyperelliptic to elliptic integrals:

"If any integral of the first species (Gattung) and the first order (Ordnung) be reducible to elliptic integrals by a rational transformation of the k^{th} degree, there can always be found a corresponding system of ϑ-moduli in which

$$\tau_{12} = \frac{1}{k}\text{."}$$

Applying to the periods the linear transformation

$$\begin{aligned} \omega_1 &= \omega_1 - \omega_4, \\ \bar{\omega}_2 &= - \omega_4, \\ \bar{\omega}_3 &= \omega_3, \\ \bar{\omega}_4 &= \omega_2 - \omega_3, \end{aligned}$$

whence follow

$$\bar{\tau}_{11} = \frac{\tau_{11}\tau_{22} - \tau_{12}^2}{\tau_{22}}, \quad \bar{\tau}_{12} = - \frac{\tau_{12} + \tau_{11}\tau_{22} - \tau_{12}^2}{\tau_{22}}, \quad \bar{\tau}_{22} = \frac{-1 + 2\tau_{12} + \tau_{11}\tau_{22} - \tau_{12}^2}{\tau_{22}};$$

we may give the relation $\tau_{12} = \frac{1}{2}$ also in the form

$$\bar{\tau}_{11} = \bar{\tau}_{22},$$

which will be useful afterwards.

§11.

An analogous proceeding being applicable to the other cases, I confine myself to giving for each of them the figure which shows the paths of integration and the value of $\frac{\sqrt{f'(z)}}{\sqrt{f(z)}}$, the linear transformation of the periods and the resulting relation between the $\tau_{\alpha\beta}$.

II. Case: $\qquad f(z) = (z^5 + 1)z.$

$$\alpha_1 = e^{\frac{-\pi i}{5}}, \ \alpha_2 = e^{\frac{\pi i}{5}}, \ \alpha_3 = e^{\frac{3\pi i}{5}}, \ \alpha_4 = -1, \ \alpha_5 = e^{\frac{F\pi i}{5}}, \ \alpha_6 = 0.$$

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* Acta Mathematica, Bd. 4, p. 400.

† Bulletin de la Société mathematique de France, Tome XI.

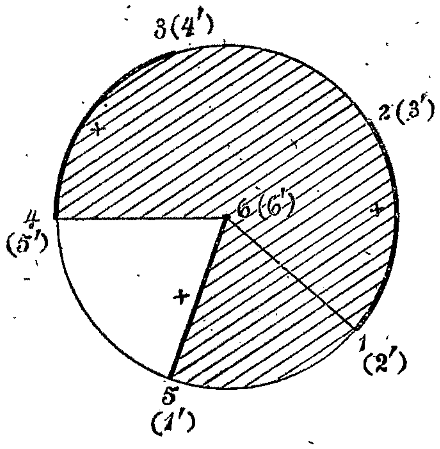


Fig. 2.

$$\varepsilon = e^{\frac{2\pi i}{5}},$$
$$z = \varepsilon z',$$
$$f(z) = \varepsilon f(z') = f'(z'),$$
$$\frac{\sqrt{f'(z)}}{\sqrt{f(z)}} = \begin{cases} +\,e^{\frac{\pi i}{5}} \text{ in the shaded portion,} \\ -\,e^{\frac{\pi i}{5}} \text{ in the blank portion.} \end{cases}$$

Thence
$$\omega_1' = \omega_3 - \omega_4,$$
$$\omega_2' = \omega_4,$$
$$\omega_3' = -\omega_1 + \omega_3,$$
$$\omega_4' = -\omega_1 - \omega_2 + \omega_3,$$
and $\tau_{11} = 1 - \varepsilon^4$, $\tau_{12} = -\varepsilon^2 - \varepsilon^4$, $\tau_{22} = \varepsilon$.

III. Case: $f(z) = z(z^4 + az^2 + 1)$.

$$\alpha_5 = 0,\ \alpha_6 = \infty,\ \alpha_3 = -\alpha_1,\ \alpha_4 = -\alpha_2,\ \alpha_1 = \frac{1}{\alpha_2},\ \alpha_3 = \frac{1}{\alpha_4}.$$

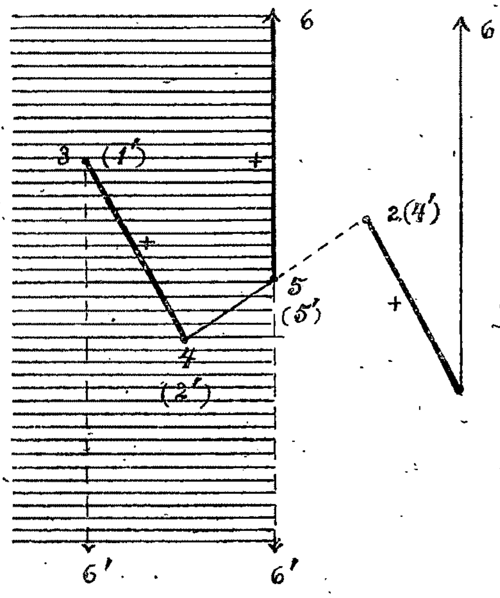


Fig. 3.

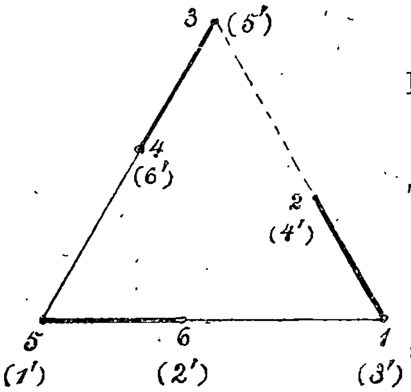
$$z = -z',$$
$$f(z) = f'(z') = -f(z'),$$
$$\frac{\sqrt{f'(z)}}{\sqrt{f(z)}} = \pm i,$$

+ in the shaded portion,
− in the blank portion.

Thence
$$\omega_1' = -\omega_2,$$
$$\omega_2' = \omega_1,$$
$$\omega_3' = -\omega_4 + \omega_1,$$
$$\omega_4' = \omega_3 - \omega_2,$$
and $\tau_{12} = \frac{1}{2}$, $\tau_{11} = \tau_{22}$.

IV. Case: $f(z) = z^6 + az^3 + 1$.

$$\left.\begin{matrix} \alpha_4 = \varepsilon\alpha_2, & \alpha_6 = \varepsilon^2\alpha_2 \\ \alpha_3 = \varepsilon\alpha_1, & \alpha_5 = \varepsilon^2\alpha_1 \end{matrix}\right\}\quad \alpha_2 = \frac{1}{\alpha_1},\ \varepsilon = e^{\frac{2\pi i}{3}}.$$



$$z = \varepsilon z',$$

$$f(z) = f'(z') = f(z').$$

Here we have in the whole plane

$$\frac{\sqrt{f'(z)}}{\sqrt{f(z)}} = +1.$$

Thence

$$\omega_1' = \omega_2,$$
$$\omega_2' = -\omega_1 - \omega_2;$$
$$\omega_3' = -\omega_3 + \omega_4;$$
$$\omega_4' = -\omega_3,$$

and the resulting relations are

$$\tau_{11} = \tau_{22} = 2\tau_{12},$$

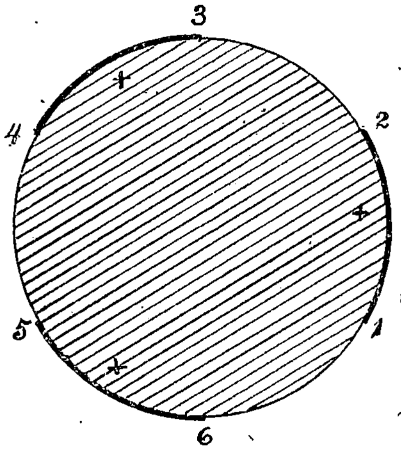
Fig. 4.

which may also be given in the form

$$\bar{\tau}_{12} = \frac{1}{2}, \; 12\bar{\tau}_{11}\bar{\tau}_{22} + 1 = 0.$$

V. Case: $f(z) = z^6 + 1$.

$$\alpha_1 = e^{\frac{-\pi i}{6}}, \; \alpha_2 = j\alpha_1, \; \alpha_3 = j^2\alpha_1, \; \alpha_4 = j^3\alpha_1, \; \alpha_5 = j^4\alpha_1, \; \alpha_6 = j^5\alpha_1.$$



$$j = e^{\frac{\pi i}{3}},$$
$$z = jz',$$
$$f(z) = f'(z') = f(z'),$$
$$\frac{\sqrt{f'(z)}}{\sqrt{f(z)}} = \pm 1,$$

+ in the shaded portion,
— in the blank portion.

Thence

$$\omega_1' = \omega_3 - \omega_4,$$
$$\omega_2' = \omega_4,$$
$$\omega_3' = -\omega_1,$$
$$\omega_4' = -\omega_1 - \omega_2.$$

Fig. 5.

The square of this transformation being identical with the transformation of Case IV, we find

$$\tau_{11} = \tau_{22} = \frac{2}{\sqrt{3}}i, \quad \tau_{12} = \frac{1}{\sqrt{3}}i;$$

or,

$$\bar{\tau}_{11} = \bar{\tau}_{22} = \frac{i}{2\sqrt{3}}; \quad \bar{\tau}_{12} = \frac{1}{2}.$$

VI. Case: $$f(z) = z(z^4+1).$$

$$\alpha_5 = 0,\ \alpha_6 = \infty,\ \alpha_1 = e^{\frac{-\pi i}{4}},\ \alpha_2 = e^{\frac{\pi i}{4}},\ \alpha_3 = e^{\frac{3\pi i}{4}},\ \alpha_4 = e^{\frac{5\pi i}{4}}.$$

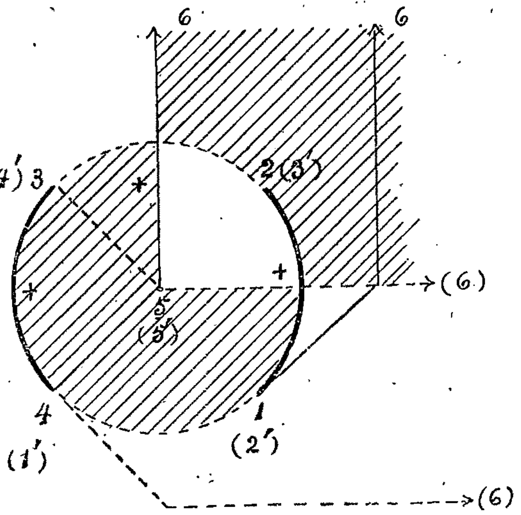


Fig. 6.

$$z = iz',$$
$$f(z) = f'(z') = if(z'),$$
$$\frac{\sqrt{f'(z)}}{\sqrt{f(z)}} = \pm e^{\frac{\pi i}{4}},$$

+ in the shaded portions,
— in the blank portions.

Thence

$$\omega_1' = \omega_3 + \omega_4,$$
$$\omega_2' = -\omega_1 + \omega_2 - \omega_3 + \omega_4,$$
$$\omega_3' = -\omega_1 - \omega_3,$$
$$\omega_4' = \omega_1 - \omega_2 + \omega_3.$$

The square of this transformation being identical with the transformation of Case III, we find $$\tau_{11} = \tau_{22} = \frac{-1+i\sqrt{2}}{2},\ \tau_{12} = \frac{1}{2}.$$

On comparison of the different cases, the relations between the $\tau_{\alpha\beta}$ for the function $z(z^4+1)$ appear as a special case of the relations for $z(z^4+\alpha z^2+1)$, the latter being themselves a special case of the relations for $z^6+\alpha z^4+\beta z^2+1$, as was to be expected from the fact that the cyclic group ($n=2$) is contained as a subgroup in the diedron group ($n=2$), and the latter in the oktaedron group. An analogous remark may be made respecting the three groups, cyclic $n=2$, diedron $n=3$, and diedron $n=6$; and respecting the three groups, cyclic $n=2$, diedron $n=2$, and diedron $n=6$.

A few words may be added concerning the sufficiency of the relations between the $\tau_{\alpha\beta}$. Taking for instance the Case V,
$$\bar{\tau}_{11} = \bar{\tau}_{22} = \frac{i}{2\sqrt{3}},\ \bar{\tau}_{12} = \frac{1}{2},$$
we find first that $\tau_{\alpha\beta}$ remain unchanged for a linear transformation whose period is 6.* Thence we infer as above, §9, that f must remain unchanged for a corresponding substitution of the variables possessing likewise the period 6, and therefore for all the substitutions of the diedron group for $n=6$, q. e. d.

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* Any operation T is said to possess the period μ, if μ is the least possible integer for which $T^{\mu}=1$.

In the case of $$\bar{\tau}_{12} = \frac{1}{2}\,, \quad 12\bar{\tau}_{11}\bar{\tau}_{22} + 1 = 0,$$

we find similarly that f remains unchanged for the substitutions of the diedron group $n = 3$; but, that F may remain unchanged for no other substitution, it is necessary to add a certain inequality, expressing the condition that $\bar{\tau}_{11}$, $\bar{\tau}_{12}$, $\bar{\tau}_{22}$ be not reducible by a linear transformation to the values
$$\bar{\tau}_{11} = \bar{\tau}_{22} = \frac{i}{2\sqrt{3}}\,, \quad \bar{\tau}_{12} = \frac{1}{2}\,,$$
which are characteristic for the diedron group $n = 6$. A similar remark is applicable to the other cases.

We conclude by combining in a table the relations which exist for the sextics with linear substitutions into themselves, on the one hand between the rational invariants, and on the other hand between the $\tau_{\alpha\beta}$.

	Canonical form.	Invariant criteria.	Relations between the $\tau_{\alpha\beta}$.
I.	$z_1^6 + \alpha z_1^4 z_2^2 + \beta z_1^2 z_2^4 + z_2^6$	$R = 0$; $A_{ll}A_{\bar{m}m} - A_{lm} \gtrless 0$	$\tau_{12} = \frac{1}{2}$
II.	$z_1(z_1^5 + z_2^5)$	$A = 0,\ B = 0,\ C = 0$; $D \gtrless 0$	$\left.\begin{array}{l}\tau_{11} = 1 - \varepsilon^4 \\ \tau_{12} = -\varepsilon^2 - \varepsilon^4 \\ \tau_{22} = \varepsilon\end{array}\right\}\ \varepsilon = e^{\frac{2\pi i}{5}}$
III.	$z_1 z_2 (z_1^4 + \alpha z_1^2 z_2^2 + z_2^4)$	$3AB^2 - 6BC + 4A^2C - 18D = 0$, $4B^3 + 5ABC + 6C^2 - 3AD = 0$; $D \gtrless 0,\ C^2 - \frac{1}{6}B^3 \gtrless 0$	$\tau_{12} = \frac{1}{2}$, $\tau_{11} = \tau_{22}$
IV.	$z_1^6 + \alpha z_1^3 z_2^3 + z_2^6$	$C^2 - \frac{1}{6}B^3 = 0$, $9D - 2B(6C + AB) = 0$; $D \gtrless 0,\ 2AB - 15C \gtrless 0$	$\tau_{12} = \frac{1}{2}$, $12\tau_{11}\tau_{22} + 1 = 0$
V.	$z_1^6 + z_2^6$	$6B - A^2 = 0,\ 6C + AB = 0$, $D = 0$; $A \gtrless 0$	$\tau_{12} = \frac{1}{2}$, $\tau_{11} = \tau_{22} = \frac{i}{2\sqrt{3}}$
VI.	$z_1 z_2 (z_1^4 + z_2^4)$	$B = 0,\ C = 0,\ D = 0$; $A \gtrless 0$	$\tau_{12} = \frac{1}{2}$, $\tau_{11} = \tau_{22} = \frac{-1 + i\sqrt{2}}{2}$

(B)

GÖTTINGEN, August, 1887.

On the Transformation of Elliptic Functions (Sequel)

By Prof. Cayley.

The chief object of the present paper is the further development of the $\rho\alpha\beta$-theory in the case $n = 7$. I recall that the forms are

$$\frac{dy}{\sqrt{1 - 2\beta y^2 + y^4}} = \frac{\rho dx}{\sqrt{1 - 2\alpha x^2 + x^4}},$$

where

$$y = \frac{x(\rho + A_2 x^2 + A_1 x^4 + x^6)}{1 + A_1 x^2 + A_2 x^4 + \rho x^6}.$$

The paragraphs are numbered consecutively with those of the former paper "On the Transformation of Elliptic Functions," vol. IX, pp. 193–224.

The Seventhic Transformation: the $\rho\alpha$-Equation. Art. Nos. 51 to 57.

51. The equation is given incorrectly Nos. 7 and 42; there was an error of sign in a term $512\alpha^3\rho$, which affected also the coefficient of $\alpha\rho$, and an error of sign in the absolute term 7. The correct form is

$$\begin{aligned} \rho^8 - 28\rho^6 - 112\alpha\rho^5 - 210\rho^4 - 224\alpha\rho^3 + (-1484 + 1344\alpha^2)\rho^2& \\ + (464\alpha - 512\alpha^3)\rho - 7 = 0&; \end{aligned}$$

or, arranging in powers of α, this is

$$\begin{aligned} &\alpha^3 . 512\rho \\ &+ \alpha^2 . -1344\rho^2 \\ &+ \alpha . 112\rho^5 + 224\rho^3 - 464\rho \\ &\quad - (\rho^8 - 28\rho^6 - 210\rho^4 - 1484\rho^2 - 7) = 0. \end{aligned}$$

This may also be written in the forms

$$\begin{aligned} (\alpha - 1)\{\alpha^2 . 512\rho + \alpha(-1344\rho^2 + 512\rho) + 112\rho^5 + 224\rho^3 - 1344\rho^2 + 48\rho\}& \\ - (\rho + 1)^7(\rho - 7) = 0&, \end{aligned}$$

and

$$\begin{aligned} (\alpha + 1)\{\alpha^2 . 512\rho + \alpha(-1344\rho^2 - 512\rho) + 112\rho^5 + 224\rho^3 + 1344\rho^2 + 48\rho\}& \\ - (\rho - 1)^7(\rho + 7) = 0&. \end{aligned}$$

To simplify the $\rho\alpha$-equation we assume $A = 8\rho\alpha - 7\rho^2$; then the $A\rho$-equation is

$$\begin{aligned} &A^3 \\ &+ A\rho^2(14\rho^4 - 119\rho^2 - 58) \\ &- \rho^2(\rho^8 - 126\rho^6 + 280\rho^4 - 1078\rho^2 - 7) = 0; \end{aligned}$$

viz., this is a cubic equation wanting its second term, and so at once solvable by Cardan's formula: say the equation is

$$A^3 + A\rho^2 q_1 - \rho^2 r_1 = 0,$$

where

$$\begin{aligned} q_1 &= 14\rho^4 - 119\rho^2 - 58, \\ r_1 &= \rho^8 - 126\rho^6 + 280\rho^4 - 1078\rho^2 - 7. \end{aligned}$$

It is convenient to recall here that, writing $\sigma = -\frac{7}{\rho}$, and $B = 8\sigma\beta - 7\sigma^2$, we have between σ, β, B precisely the same equations as between ρ, α, A; $\rho = 1$ gives $\sigma = -7$, and we have as corresponding values $\alpha = -1$, $A = -15$, $\beta = -1$, $B = -287$: these are very convenient for verification of the formulæ. Similarly $\rho = -7$ gives $\sigma = 1$, and then $\alpha = -1$, $A = -287$, $\beta = -1$, $B = -15$; but I have in general used the former values only.

52. We have $$A = f + g,$$
where

$$\begin{aligned} 3fg &= -\rho^2 q_1, \\ f^3 + g^3 &= \rho^2 r_1, \end{aligned}$$

and thence $$f^3 - g^3 = \rho^2\sqrt{r_1^2 + \frac{4\rho^2 q_1^3}{27}}.$$

We have identically

$$\begin{aligned} &27(\rho^8 - 126\rho^6 + 280\rho^4 - 1078\rho^2 - 7)^2 + 4\rho^2(14\rho^4 - 119\rho^2 - 58)^3 \\ &\quad = (\rho^6 + 75\rho^4 - 141\rho^2 + 1)^2(27\rho^4 + 122\rho^2 + 1323) \end{aligned}$$

[$\rho = 1$, this is $27.930^2 + 4(-163)^3 = 64^2.1472$; that is, $23352300 - 17322988 = 6029312$, which is right]; but it is convenient to divide by 27, so as instead of $27\rho^4 + 122\rho^2 + 1323$ to have in the formulæ $\rho^4 + \frac{122}{27}\rho^2 + 49$, or say

$$\rho^4 + K\rho^2 + 49 \left(K = \frac{122}{27}\right).$$

Hence writing

$$\begin{aligned} t_1 &= \rho^6 + 75\rho^4 - 141\rho^2 + 1, \\ \delta &= \rho^4 + K\rho^2 + 49, \end{aligned}$$

we have $$r_1^2 + \frac{4}{27}\rho^2 q_1 = t_1^2\delta,$$

and consequently
$$2f^3 = \rho^2 (r_1 + t_1\sqrt{\delta}),$$
$$2g^3 = \rho^2 (r_1 - t_1\sqrt{\delta}).$$

53. It was easy to foresee that the cube root of $r_1 \pm t_1\sqrt{\delta}$ would break up into the form $(U \pm \sqrt{\delta})\sqrt[3]{W \pm \sqrt{\delta}}$, and I was led to the actual expressions by the identities

$$20\,(14\rho^4 - 119\rho^2 - 58) = (19\rho^2 - 53)^2 - 3\,(27\rho^4 + 122\rho^2 + 1323);$$

that is, $20q_1 = (19\rho^2 - 53)^2 - 81\delta,$

and
$$27\,(\rho^2 - 7)^2 - (27\rho^4 + 122\rho^2 + 1323) = -\ 500\rho^2,$$
$$27\,(\rho^2 + 7)^2 - (27\rho^4 + 122\rho^2 + 1323) = \quad 256\rho^2;$$

or, as these may be written,

$$(\rho^2 - 7)^2 - \delta = -\frac{500}{27}\rho^2,\ \ (\rho^2 + 7)^2 - \delta = \frac{256}{27}\rho^2.$$

We in fact have further the two identities

$$\begin{aligned} 1000\,(\rho^6 + 75\rho^4 - 141\rho^2 + 1) \\ &= \{(19\rho^2 - 53)^3 + 243\,(19\rho^2 - 53)(\rho^4 + K\rho^2 + 49)\} \\ &+ \{27\,(19\rho^2 - 53)^2 \qquad + 729\,(\rho^4 + K\rho^2 + 49)\}(-\rho^2 + 7), \end{aligned}$$

$$\begin{aligned} -1000\,(\rho^8 - 126\rho^6 + 280\rho^4 - 1078\rho^2 - 7) \\ &= \{(19\rho^2 - 53)^3 + 243\,(19\rho^2 - 53)\,(\rho^4 + K\rho^2 + 49)\}(-\rho^2 + 7) \\ &+ \{27\,(19\rho^2 - 53)^2 \qquad + 729\,(\rho^4 + K\rho^2 + 49)\}(\rho^4 + K\rho^2 + 49), \end{aligned}$$

viz., writing $19\rho^2 - 53 = 9U,\ -\rho^2 + 7 = W,$
these equations become

$$\frac{1000}{729}t_1 = U^3 + 3U\delta + (3U^2 + \delta)\,W,$$
$$-\frac{1000}{729}r_1 = (U^3 + 3U\delta)\,W + (3U^2 + \delta)\,\delta,$$

and we have thus

$$-\frac{1000}{729}(r_1 - t_1\sqrt{\delta}) = (U + \sqrt{\delta})^3(W + \sqrt{\delta}),$$

and the like equation with $-\sqrt{\delta}$ in place of $\sqrt{\delta}$.

54. In part verification of the last-mentioned identities, observe that in the first of them, putting $\rho = 1$, and comparing first the coefficients of ρ^6 and then the coefficients of ρ^0, we ought to have

$$1000 = 19^3 + 243.19 - (27.19^2 + 729),\ = 11476 - 10476,$$
$$1000 = (-53^3 - 243.53.49) + (27.53^2 + 729.49)\,7,\ = -779948 + 780948,$$

which are right; and similarly in the second equation, comparing first the coefficients of ρ^8 and next those of ρ^0, we have

$$\begin{aligned} -1000 &= (19^3 + 243.19)(-1) + (27.19^2 + 729), = -11476 + 10476, \\ +7000 &= (-53^3 - 243.53.49)(7) + (27.53^2 + 729.49)\,49, \\ &= -5459636 + 5466636, \end{aligned}$$

which are right.

55. We have now $A = f + g$, where

$$f = -\frac{9}{10}(U - \sqrt{\delta})\sqrt[3]{\frac{1}{2}\rho^2(W - \sqrt{\delta})},$$

$$g = -\frac{9}{10}(U + \sqrt{\delta})\sqrt[3]{\frac{1}{2}\rho^2(W + \sqrt{\delta})},$$

(where observe that, multiplying these two values, we have

$$\begin{aligned} fg = \frac{81}{100}(U^2 - \delta)\sqrt[3]{\frac{1}{4}\rho^4.(W^2 - \delta)}, &= \frac{81}{100}(U^2 - \delta)\sqrt[3]{\frac{1}{4}\rho^4.\frac{-500}{27}\rho^2}, \\ &= \frac{81}{100}(U^2 - \delta)\left(-\frac{5}{3}\rho^2\right); \end{aligned}$$

that is,

$$fg = -\frac{27}{20}\rho^2(U^2 - \delta), = -\frac{27}{20}\rho^2.\frac{20q_1}{81} = -\frac{1}{3}\rho^2 q_1,$$

which is right). Or, finally, substituting for U, W, δ their values, we have, for the solution of the $A\rho$-equation, $A = f + g$, where

$$f = -\frac{9}{10}(19\rho^2 - 53 - \sqrt{\rho^4 + K\rho^2 + 49})\sqrt[3]{\frac{1}{2}\rho^2\left\{-\rho^2 + 7 - \sqrt{\rho^4 + K\rho^2 + 49}\right\}}, \left(K = \frac{122}{27}\right),$$

$$g = -\frac{9}{10}(19\rho^2 - 53 + \sqrt{\rho^4 + K\rho^2 + 49})\sqrt[3]{\frac{1}{2}\rho^2\left\{-\rho^2 + 7 + \sqrt{\rho^4 + K\rho^2 + 49}\right\}}.$$

56. In the case $\rho = 1$, α has a value $= -1$, giving for A, $= 8\rho\alpha - 7\rho^2$, the value -15; and, in fact, here $\rho^2 = 1$, and the $A\rho$-equation becomes

$$A^3 - 163A + 930 = 0,$$

that is,

$$(A + 15)(A^2 - 15A + 62) = 0,$$

the roots thus being

$$A = -15, \quad A = \frac{1}{2}(15 \pm i\sqrt{23}).$$

To verify in this case the values given by the solution of the cubic equation, observe that for $\rho^2 = 1$ we have $\delta = 50 + \frac{122}{27}$, $= \frac{1472}{27}$, and therefore

$\sqrt{\delta} = \frac{8\sqrt{23}}{3\sqrt{3}},\ = \frac{8\sqrt{69}}{9}$; also, $U = \frac{19\rho^2 - 53}{9},\ = \frac{-34}{9}$, and $W = -\rho^2 + 7,\ = 6$. Hence $U + \sqrt{\delta} = \frac{-34 + 8\sqrt{69}}{9}$, and

$$\sqrt[3]{\frac{1}{2}\rho^2(W + \sqrt{\delta})} = \sqrt[3]{3 + \frac{8\sqrt{69}}{9}},\ = \frac{\sqrt[3]{81 + 12\sqrt{69}}}{3};$$

hence

$$g = -\frac{9}{10}\,\frac{2(-17 + 4\sqrt{69})}{9}\,\frac{1}{3}\sqrt[3]{81 + 12\sqrt{69}},\ = \frac{1}{15}(17 - 4\sqrt{69})\sqrt[3]{81 + 12\sqrt{69}};$$

but the cube root is $= \frac{1}{2}(3 + \sqrt{69})$, and we have $(17 - 4\sqrt{69})(3 + \sqrt{69})$ $= -225 + 5\sqrt{69},\ = 5(-45 + \sqrt{69})$; that is, $g = \frac{1}{6}(-45 + \sqrt{69})$. Similarly $f = \frac{1}{6}(-45 - \sqrt{69})$. We have thus the real root $f + g = -15$, and the imaginary roots $f\omega + g\omega^2$ or $f\omega^2 + g\omega$, $= -\frac{15}{2}(\omega + \omega^2) + \frac{1}{6}\sqrt{69}(\omega - \omega^2)$, viz., the first term is $= \frac{15}{2}$ and the second is $\pm\frac{1}{6}\sqrt{69}.i\sqrt{3},\ = \pm\frac{1}{2}i\sqrt{23}$; thus the roots are $\frac{1}{2}(15 \pm i\sqrt{23})$, as they should be.

57. I found, by considerations arising out of the new theory Nos. 72 *et seq.*, that writing for shortness $m = i\sqrt{3}$, then, for $\rho = m - 2$, the $\rho\alpha$-equation has a root $\alpha = m$; the corresponding values of $A_1\rho^2$ thus are $A = 12m - 31$, $\rho^2 = -4m + 1$, viz., substituting this value for ρ^2 in the $A\rho$-equation, there should be a root $A = 12m - 31$. The equation becomes

$$A^3 + A(3704m - 7653) + 148306m + 206162 = 0,$$

or, as this may be written,

$$(A - 12m + 31)\{A^2 + A(12m - 31) + 2960m + 4062\} = 0,$$

and the roots thus are

$$A = 12m - 31,$$
$$A = -6m + \frac{31}{2} \pm \frac{1}{2}\sqrt{-12584m - 16777},$$

where the square root is not expressible as a rational function of m.

Expression of β as a Rational Function of α, ρ. Art. Nos. 58 to 66.

58. Writing $\sigma = -\frac{7}{\rho}$, we have β the same function of σ that ρ is of α; hence if $B = 8\sigma\beta - 7\sigma^2$, the $B\sigma$-equation is

$$\begin{aligned}&B^3\\ &+ B\sigma^2(14\sigma^4 - 119\sigma^2 - 58)\\ &- \sigma^2(\sigma^8 - 126\sigma^6 + 280\sigma^4 - 1078\sigma^2 - 7) = 0,\end{aligned}$$

and the expression for B in terms of σ is obtained from that of A by the mere change of ρ into σ. Say we have $B = f' + g'$ where

$$f' = -\frac{9}{10}(U' - \sqrt{\delta'})\sqrt[3]{\frac{1}{2}\sigma^2(W' - \sqrt{\delta'})},$$

$$g' = -\frac{9}{10}(U' + \sqrt{\delta'})\sqrt[3]{\frac{1}{2}\sigma^2(W' + \sqrt{\delta'})};$$

then we have

$$\begin{aligned}\frac{1}{2}\sigma^2(W' + \sqrt{\delta'}) &= \frac{1}{2}\frac{49}{\rho^2}\left(-\frac{49}{\rho^2} + 7 + \sqrt{\frac{2401}{\rho^4} + \frac{49K}{\rho^2} + 49}\right)\\ &= -\frac{1}{2}\cdot\frac{343}{\rho^4}(-\rho^2 + 7 - \sqrt{\rho^4 + K\rho^2 + 49})\\ &= -\frac{343}{\rho^6}\cdot\frac{1}{2}\rho^2(W - \sqrt{\delta}),\end{aligned}$$

or say $$\sqrt[3]{\frac{1}{2}\sigma^2(W' + \sqrt{\delta'})} = -\frac{7}{\rho^2}\sqrt[3]{\frac{1}{2}\rho^2(W - \sqrt{\delta})};$$

and similarly $$\sqrt[3]{\frac{1}{2}\sigma^2(W' + \sqrt{\delta'})} = -\frac{7}{\rho^2}\sqrt[3]{\frac{1}{2}\rho^2(W + \sqrt{\delta})}.$$

The cube roots which enter into the expression of B are thus identical with those in the expression of A, and it hence appears that B can be expressed rationally in terms of A, ρ; or, what is the same thing, β can be expressed rationally in terms of α, ρ.

59. The *à priori* reason is obvious: the $\rho\alpha$-equation is a cubic in α, but of the order 8 in ρ; hence to a given value of α there correspond 8 values of ρ. Similarly the $\sigma\beta$-equation is a cubic in β, but of the order 8 in σ, or if for σ we substitute its value $= -\frac{7}{\rho}$, then we have a $\rho\beta$-equation which is a cubic in β, but of the order 8 in ρ. In the absence of any special relation between this $\rho\beta$-equation and the $\rho\alpha$-equation, there would correspond to each of the 8 values of ρ, 3 values of β; that is, to a given value of α there would correspond $8 \times 3, = 24$ values of β. But, in fact, to a given value of α there correspond

only 8 values of β, and the two cubic equations are related to each other in such wise that this is so; viz., the relation between them is such that it is possible by means of them to express β as a rational function of ρ, α.

60. Returning to the investigation, we have

$$9U' = 19\sigma^2 - 53, = \frac{19.49}{\rho^2} - 53;$$

or, writing $63\overline{U} = 53\rho^2 - 931$,

this is $\qquad U' = -\frac{7}{\rho^2}\overline{U}$, whence $U' \pm \sqrt{\delta'} = -\frac{7}{\rho^2}(\overline{U} \mp \sqrt{\delta})$.

Hence writing

$$\theta = \sqrt[3]{\frac{1}{2}\rho^2(W - \sqrt{\delta})},\ \phi = \sqrt[3]{\frac{1}{2}\rho^2(W + \sqrt{\delta})},$$

we have

$$f = -\frac{9}{10}(U - \sqrt{\delta})\,\theta,\ f' = -\frac{9}{10}\frac{49}{\rho^4}(\overline{U} + \sqrt{\delta})\,\phi,$$

$$g = -\frac{9}{10}(U + \sqrt{\delta})\,\phi,\ g' = -\frac{9}{10}\frac{49}{\rho^4}(\overline{U} - \sqrt{\delta})\,\theta,$$

so that, putting for shortness

$$L = -\frac{9}{10}(U - \sqrt{\delta}),\ \overline{L} = -\frac{9}{10}\frac{49}{\rho^4}(\overline{U} - \sqrt{\delta}),$$

$$M = -\frac{9}{10}(U + \sqrt{\delta}),\ \overline{M} = -\frac{9}{10}\frac{49}{\rho^4}(\overline{U} + \sqrt{\delta}),$$

we have

$$A = L\theta + M\phi,\ B = \overline{L}\theta + \overline{M}\phi,$$

where θ^3, ϕ^3 and $\theta\phi$ are each of them free from any cube root; we have, in fact,

$$\theta\phi = \sqrt[3]{\frac{1}{4}\rho^4(W^2 - \delta)},\ = \sqrt[3]{\frac{1}{4}\rho^4 . \frac{-500}{27}\rho^2},\ = -\frac{5}{3}\rho^2,$$

and it may be added that

$$\begin{aligned} 3LM\theta\phi &= -\rho^2 q_1,\ \text{whence } LM = \frac{1}{5}q_1, \\ L^3\theta^3 + M^3\phi^3 &= \quad \rho^2 r_1, \\ L^3\theta^3 - M^3\phi^3 &= \quad \rho^2 t_1\sqrt{\delta}; \end{aligned}$$

these are, in fact, only the equations obtained by writing $L\theta$, $M\phi$ in place of f, g respectively.

61. In the case $\rho = 1$ we have $\sigma = -7$, the equation for B becomes

$$B^3 + 1358525B + 413536578 \qquad = 0;$$

that is, $\qquad (B + 287)(B^2 - 287B + 1440894) = 0,$

and the roots are

$$-287 \text{ and } \frac{1}{2}(287 \pm 497i\sqrt{23}),\ \text{or, say } -7.41 \text{ and } \frac{7}{2}(41 \pm 71i\sqrt{23}).$$

We have as before, $\sqrt{\delta} = \frac{8\sqrt{69}}{9}$, and $\sqrt[3]{\frac{1}{2}\rho^2(W+\sqrt{\delta})} = \frac{1}{3}\sqrt[3]{81+12\sqrt{69}} = \theta$; also, $\overline{U} = \frac{-878}{63}$, whence $\overline{U}+\sqrt{\delta} = \frac{2(-439+28\sqrt{69})}{63}$. We thus have

$$\begin{aligned} f' &= -\frac{9}{10}\,49\,.\,\frac{2(-439+28\sqrt{69})}{63}\,\frac{1}{3}\sqrt[3]{81+12\sqrt{69}}, \\ &= -\frac{7}{15}(-439+28\sqrt{69})\sqrt[3]{81+12\sqrt{69}}, \end{aligned}$$

or, putting for the cube root its value $= \frac{1}{2}(3+\sqrt{69})$, this is

$$f' = -\frac{7}{30}(-439+28\sqrt{69})(3+\sqrt{69}), = -\frac{287}{2}+\frac{497}{6}\sqrt{69}.$$

Similarly $g' = -\frac{287}{2}-\frac{497}{6}\sqrt{69}$; and forming the values $f'+g'$, $\omega f'+\omega^2 g'$, $\omega^2 f'+\omega g'$, we have the real root -287 and the imaginary roots $\frac{1}{2}(287 \pm 497 i\sqrt{23})$, as above.

62. We have the equations

$$\begin{aligned} B &= \overline{L}\theta + \overline{M}\phi, \\ A &= L\theta + M\phi, \\ A^2 - 2LM\theta\phi &= \frac{M^2\phi^3}{\theta\phi}\,\theta + \frac{L^2\theta^3}{\theta\phi}\,\phi, \end{aligned}$$

from which, eliminating θ, ϕ so far as they present themselves linearly on the right-hand side, and in the resulting equation replacing $\theta\phi$ and $LM\theta\phi$ by their values, we have

$$\begin{vmatrix} B, & \overline{L}, & \overline{M} \\ A, & L, & M \\ -\frac{5}{3}\rho^2\left(A^2+\frac{2}{3}\rho^2 q_1\right), & M^2\phi^3, & L^2\theta^3 \end{vmatrix} = 0;$$

that is,

$$B(L^3\theta^3 - M^3\phi^3) = A(L^2\overline{L}\theta^3 - M^2\overline{M}\phi^3) - \frac{5}{3}\rho^2\left(A^2+\frac{2}{3}\rho^2 q_1\right)(L\overline{M}-\overline{L}M).$$

This may be written

$$\begin{aligned} B\rho^2 t_1\sqrt{\delta} = A\Big\{-\frac{729}{1000}\frac{49}{\rho^4}\Big[&(U-\sqrt{\delta})^2(\overline{U}-\sqrt{\delta})\,\frac{1}{2}\rho^2(W-\sqrt{\delta}) \\ &-(U+\sqrt{\delta})^2(\overline{U}+\sqrt{\delta})\,\frac{1}{2}\rho^2(W+\sqrt{\delta})\Big]\Big\} \\ -\frac{5}{3}\rho^2\left(A^2+\frac{2}{3}\rho^2 q_1\right)\frac{81}{100}\frac{49}{\rho^4}\Big[&(U-\sqrt{\delta})(\overline{U}+\sqrt{\delta}) \\ &-(U+\sqrt{\delta})(\overline{U}-\sqrt{\delta})\Big], \end{aligned}$$

where the terms in [ ] contain each of them the factor $\surd\delta$. Omitting this factor from the equation, and multiplying by ρ^2, we have

$$B\rho^4 t_1 = \frac{81}{100}\,49\left\{\frac{9}{10}A\left[(U^2+2U\overline{U}+\delta)\,W+U^2\overline{U}+(2U+\overline{U})\,\delta\right]\right.$$
$$\left.-\frac{10}{3}\left(A^2+\frac{2}{3}\rho^2 q_1\right)(U-\overline{U})\right\},$$

which I verify at this stage by writing as before, $\rho = 1$. We have $B = -287$, $A = -15$, $t_1 = -64$, $q_1 = -163$, $W = 6$, $U = -\frac{34}{9}$, $\overline{U} = \frac{-878}{63}$; and, omitting intermediate steps, the equation becomes

$$287.64 = \frac{81.49}{100}\left(\frac{2496000}{567} - \frac{2233600}{567}\right), = \frac{81.49}{100.567}\,262400, = 18368,$$

which is right.

63. We require the values of $(U^2+2U\overline{U}+\delta)\,W+U^2\overline{U}+(2U+\overline{U})\,\delta$, and of $U-\overline{U}$: I insert some of the steps of the calculation. We have

$$U^2+2U\overline{U}+\delta = \frac{1}{63^2}\{(133\rho^2-371)(239\rho^2-2233)+63^2(\rho^4+49)+3.49.122\rho^2\}$$
$$= \frac{1}{63^2}\{35756\rho^4-367724\rho^2+1022924\}$$
$$= \frac{4}{567}\{\ 1277\rho^4-\ 13133\rho^2+\ \ 36533\}.$$

Multiplying by W, $= -\rho^2+7$, we have

$$(U^2+2U\overline{U}+\delta)\,W = \frac{4}{567}\{-\ 1277\rho^6+\ 22072\rho^4-\ 128464\rho^2+255731\}$$
$$= \frac{4}{5103}\{-11493\rho^6+198648\rho^4-1156176\rho^2+2301579\}$$

$$U^2\overline{U} = \frac{1}{81.63}(19\rho^2-53)^2(53\rho^2-931)$$
$$= \frac{1}{5103}\quad\{19133\rho^6-442833\rho^4+2023911\rho^2-2615179\},$$

$$(2U+\overline{U})\,\delta = \frac{1}{63.27}(319\rho^2-1673)(27\rho^4+122\rho^2+1323)$$
$$= \frac{1}{1701}\{8613\rho^6-6253\rho^4+217931\rho^2-221379\}$$
$$= \frac{1}{5103}\quad\{25839\rho^6-\ \ 18759\rho^4+\ \ 653793\rho^2-6640137\},$$

whence

$$U^2\overline{U}+(2U+\overline{U})\,\delta = \frac{1}{5103}\quad\{44972\rho^6-461592\rho^4+2677704\rho^2-9255316\}$$
$$= \frac{4}{5103}\quad\{11243\rho^6-115398\rho^4+\ 669426\rho^2-2313829\}.$$

Hence, adding, we obtain

$$
\begin{aligned}
(U^2+2U\overline{U}+\delta)\,W+U^2\overline{U}+(2U+\overline{U})\,\delta & \\
&= \frac{4}{5103}\,\{-250\rho^6+83250\rho^4-486750\rho^2-12250\} \\
&= \frac{-1000}{5103}\,\{\rho^6-333\rho^4+1947\rho^2+49\}:
\end{aligned}
$$

and we have at once

$$U-\overline{U}=\frac{1}{63}\,(80\rho^2+560)=\frac{80}{63}\,(\rho^2+7).$$

64. We now find

$$
\begin{aligned}
B\rho^4 t_1 = & -7A\,(\rho^6-333\rho^4+1947\rho^2+49) \\
& -56\,(3A^2+2\rho^2 q_1)(\rho^2+7),
\end{aligned}
$$

viz. substituting for t_1, q_1 their values, this is

$$
\begin{aligned}
B\rho^4\,(\rho^6+75\rho^4-141\rho^2+1) = & -7A\,(\rho^6-333\rho^4+1947\rho^2+49) \\
& -56\,\big(3A^2+2\rho^2\,(14\rho^4-119\rho^2+1)\big)(\rho^2+7),
\end{aligned}
$$

which is the value of B, expressed rationally in terms of ρ, A; it will be observed that B is obtained as a quadric function of A, which is the proper form.

Writing $\rho=-1$, we have $A=-15$, $B=-287$, $t_1=-64$, $q_1=-163$; and the equation is

$$287\,.\,64=105\,.\,1664-56\,.\,349\,.\,8,\ =174720-156352,\ =18368,$$

which is right.

65. Writing for B, A their values $=-\dfrac{56}{\rho}\,\beta-\dfrac{343}{\rho^2}$, and $8\rho\alpha-7\rho^2$, we have

$$
\begin{aligned}
\rho^4\left(-\frac{56}{\rho}\,\beta-\frac{343}{\rho^2}\right)t_1 = & \ (-56\rho\alpha+49\rho^2)(\rho^6-333\rho^4+1947\rho^2+49) \\
& -56\,(192\rho^2\alpha^2-336\rho^3\alpha+147\rho^4+2\rho^2 q_1)(\rho^2+7);
\end{aligned}
$$

that is,

$$
\begin{aligned}
-56\rho^3\beta t_1 = & -56\,.\,192\rho^2\,(\rho^2+7)\,\alpha^2 \\
& -56\rho\alpha\,(\rho^6-333\rho^4+1947\rho^2+49) \\
& +56\,.\,336\rho^3\alpha\,(\rho^2+7) \\
& +49\rho^2\,(\rho^6-333\rho^4+1947\rho^2+49) \\
& -56\,\big(147\rho^4+2\rho^2\,(14\rho^4-119\rho^2-58)\big)(\rho^2+7) \\
& +343\rho^2\,(\rho^6+75\rho^4-141\rho^2+1),
\end{aligned}
$$

where the fourth and sixth lines unite into a term divisible by 56, viz. omitting in the first instance a factor 49, the lines are

$$\rho^8-333\rho^6+1947\rho^4+49\rho^2$$

and

$$7\rho^6+525\rho^6-987\rho^4+7\rho^2,$$

which together are $= 8\rho^8 + 192\rho^6 + 960\rho^4 + 56\rho^2$,
and hence, restoring the factor 49, the lines are

$$= 392\,(\rho^8 + 24\rho^6 + 120\rho^4 + 7\rho^2),$$

and the formula now easily becomes

$$\begin{aligned}\rho^2\beta t_1 = {} & 192\rho\,(\rho^2 + 7)\,\alpha^2 \\ & + (\rho^6 - 669\rho^4 - 405\rho^2 + 49)\,\alpha \\ & + \rho\,(21\rho^6 - 63\rho^4 - 1593\rho^2 - 861),\end{aligned}$$

where the last line is

$$= \rho\,(\rho^2 + 7)(21\rho^4 - 210\rho^2 - 123).$$

66. Hence, finally, substituting for t_1 its value, we have

$$\begin{aligned}\beta\rho^2\,(\rho^6 + 75\rho^4 - 141\rho^2 + 1) = {} & 3\rho\,(\rho^2 + 7)(64\alpha^2 + 7\rho^4 - 70\rho^2 - 41) \\ & + \alpha\,(\rho^6 - 669\rho^4 - 405\rho^2 + 49),\end{aligned}$$

which is the expression for β as a rational function of ρ, α.

Here $\rho = 1$, $\alpha = -1$, $\beta = -1$ give $64 = -960 + 1024$, which is right, and again $\rho = -7$, $\alpha = -1$, $\beta = -1$ give

$$\begin{aligned}-49\,(117649 + 180075 - 6909 + 1) = {} & -21.56\,(64 + 16807 - 3430 - 41) \\ & -(117649 - 1606269 - 19845 + 49);\end{aligned}$$

that is $\quad -49.290816 = -1176.13400 + 1508416$,
or $\quad -14249984 = -15758400 + 1508416$, which is right.

The αβ-Differential Equation. Art. No. 67.

67. We have, No. 10,
$$\frac{d\beta}{\beta^2 - 1} = \frac{\rho^2}{7}\,\frac{d\alpha}{\alpha^2 - 1},$$
and it should of course be possible to verify this equation by means of the $\rho\alpha$-equation and the value just obtained for β. But the expression for $\dfrac{d\rho}{d\alpha}$ given by the $\rho\alpha$-equation is of so complicated a form that I do not see in what way the verification will come out, and I have not attempted to effect it.

The Coefficients A_1 and A_2. Art. Nos. 68 to 71.

68. These are given by the formulæ No. 47, viz. we have

$$\begin{aligned}A_1 &= \frac{1}{\rho}\,7\,(\alpha^2 - 1)\,\frac{d\rho}{d\alpha} - \frac{7}{2}\,\alpha + \frac{1}{2}\,\beta\rho^2, \\ A_2 &= 7\,(\alpha^2 - 1)\,\frac{d\rho}{d\alpha} - \frac{19}{6}\,\alpha\rho + \frac{1}{6}\,\beta\rho^3,\end{aligned}$$

where $\frac{d\rho}{d\alpha}$ and β have each of them to be expressed in terms of ρ, α; we have thus A_1 and A_2 each of them expressible rationally in terms of ρ, α; but I have not attempted to effect the substitutions.

69. The five equations of No. 42, merely collecting the terms, are

$$\begin{gathered}
12A_2 - 6A_1^2 - 8\alpha A_1 + \rho^4 - 7 = 0, \\
(-6A_1 - 32\alpha + 2\rho^3)A_2 - 2A_1^3 - 8A_1 + 30\rho = 0, \\
(\rho^2 - 4)A_2^2 + (-4A_1^2 - 8\alpha A_1 + 6)A_2 - 5A_1^2 + (2\rho^3 + 4\rho)A_1 - 72\alpha\rho = 0, \\
-2A_1A_2^2 + \{(2\rho^2 - 4)A_1 - 6\rho\}A_2 - 4\rho A_1^2 - 32\rho\alpha A_1 + 2\rho^3 + 28\rho = 0, \\
-3A_2^2 + (-4\rho A_1 + 2\rho^2 - 8\alpha\rho)A_2 + \rho^2 A_1^2 + 10\rho A_1 - 6\rho^2 = 0,
\end{gathered}$$

which would of course be all of them satisfied by the values of A_1, A_2 as rational functions of ρ, α, viz. the substitution of these values in any one of the equations would give a function of ρ, α containing as a factor the expression on the left-hand side of the $\rho\alpha$-equation.

70. Or again, the equations should determine A_1 and A_2 as rational functions of ρ, α, but there is no obvious way of finding such values in a simple form. We of course have $$12A_2 = 6A_1^2 + 8\alpha A_1 - \rho^4 + 7,$$ and using this value to eliminate A_2 from the remaining equations we find the following four equations:

$$\begin{aligned}
&A_1^3.30 + A_1^2(120\alpha - 6\rho^3) + A_1\{128\alpha^2 - 8\rho^3\alpha - 3\rho^4 + 69\} \\
&\qquad + \alpha(-16\rho^4 + 112) + \rho^7 - 7\rho^3 - 180\rho = 0, \\
&A_1^4(36\rho^2 - 432) + A_1^3\alpha(96\rho^2 - 1344) \\
&\qquad + A_1^2\{\alpha^2(64\rho^2 - 1024) - 12\rho^6 + 48\rho^4 + 84\rho^2 - 624\} \\
&\qquad + A_1\{\alpha(-16\rho^6 + 160\rho^4 + 112\rho^2 - 544) + 288\rho^3 + 576\rho\} \\
&\qquad + \{-10368\alpha\rho + \rho^{10} - 4\rho^8 - 14\rho^6 - 16\rho^4 + 49\rho^2 - 308\} = 0, \\
&A_1^5.36 + A_1^4.96\alpha + A_1^3\{64\alpha^2 - 12\rho^4 - 72\rho^2 + 208\} \\
&\qquad + A_1^2\{\alpha(-16\rho^4 - 96\rho^2 + 304) + 504\rho\} \\
&\qquad + A_1\{\alpha.2592\rho + \rho^8 - 12\rho^6 + 10\rho^4 + 84\rho^2 - 119\} \\
&\qquad + 36\rho^5 - 144\rho^3 - 2268\rho = 0, \\
&A_1^4.36 + A_1^3(96\alpha + 96\rho) + A_1^2\{64\alpha^2 + 320\alpha\rho - 12\rho^4 - 96\rho^2 + 84\} \\
&\qquad + A_1\{256\alpha^2\rho + \alpha(-80\rho^4 + 112) - 16\rho^5 - 368\rho\} \\
&\qquad + \{\alpha(-32\rho^5 + 224\rho) + \rho^8 + 8\rho^6 - 14\rho^4 + 232\rho^2 + 49\} = 0,
\end{aligned}$$

and we could from these equations obtain various rational expressions for A_1 and its powers, but these would apparently be of degrees far too high in ρ and α.

71. It is to be remarked that for $\rho = 1$, $\alpha = -1$, the values of A_1, A_2 are $A_1 = A_2 = 3$, viz. these belong to the solution

$$y = \frac{x(1+3x^2+3x^4+x^6)}{1+3x^2+3x^4+x^6}, = x, \text{ of } \frac{dy}{1+y^2} = \frac{dx}{1+x^2};$$

and that for $\rho = -7$, $\alpha = -1$, the values are $A_1 = -21$, $A_2 = 35$, viz. these belong to the solution

$$y = \frac{-7x+35x^3-21x^5+x^7}{1-21x^2+35x^4-7x^6} \text{ of } \frac{dy}{1+y^2} = \frac{-7dx}{1+x^2}.$$

For example, the equation $12A_2 = 6A_1^2 + 8\alpha A_1 - \rho^4 + 7$ becomes, for the first set of values, $36 = 54 - 24 - 1 + 7$, and for the second set of values, $420 = 2646 + 168 - 2401 + 7$, which are each of them right.

New Form of the Seventhic Transformation. Art. Nos. 72 to 83.

72. For the quartic function $1 - 2\alpha x^2 + x^4$, the coefficients a, b, c, d, e are $= 1, 0, -\frac{1}{3}\alpha, 0, 1$, and hence the invariants I, J and the discriminant Δ are

$$I = 1 + \frac{1}{3}\alpha^2, = \frac{1}{3}(\alpha^2+3),$$

$$J = -\frac{1}{3}\alpha + \frac{1}{27}\alpha^3, = \frac{1}{27}(\alpha^3 - 9\alpha),$$

$$\Delta = I^3 - 27J^2, = \frac{1}{27}\{(\alpha^2+3)^3 - (\alpha^3-9\alpha)^2\}, = (\alpha^2-1)^2, \text{ whence } \sqrt[12]{\Delta} = \sqrt[6]{\alpha^2-1}.$$

This being so, then assuming

$$\rho = p\frac{\sqrt[6]{\alpha^2-1}}{\sqrt[6]{\beta^2-1}},$$

the differential equation

$$\frac{dy}{\sqrt{1-2\beta y^2+y^4}} = \frac{\rho dx}{\sqrt{1-2\alpha x^2+x^4}}$$

becomes

$$\frac{\sqrt[6]{\beta^2-1}\,dy}{\sqrt{1-2\beta y^2+y^4}} = \frac{p\sqrt[6]{\alpha^2-1}\,dx}{\sqrt{1-2\alpha x^2+x^4}},$$

viz. this is, for the radicals $\sqrt{1-2\alpha x^2+x^4}$ and $\sqrt{1-2\beta y^2+y^4}$, the form considered by Klein in the paper "Ueber die Transformation der Elliptischen Fonctionen und die Auflösung der Gleichungen fünften Grades," Math. Ann., t. XIV (1879), pp. 111–172. I notice that there is some error as to a factor 7, and that p is equal to the z of p. 148, not as might appear $= \frac{1}{7}z$.

73. The modular equation presents itself in the form given p. 143, viz. this is

$$\mathbf{J}:\mathbf{J}-1:1=(\tau^2+13\tau+49)(\tau^2+5\tau+1)^3:(\tau^4+14\tau^3+63\tau^2+70\tau-7)^2:1728\tau,$$

with the like relation in $\mathbf{J}'$, τ', and then $\tau\tau'=49$. We have thus $\mathbf{J}$, $\mathbf{J}'$ each given as a function of τ, and thence by elimination of τ we have the modular equation as a relation between the absolute invariants $\mathbf{J}$, $\mathbf{J}'$. But $\tau=p^2$, and for the form $1-2\alpha x^2+x^4$, as appears above, we have

$$\mathbf{J}-1,=\frac{27J^2}{\Delta};=\frac{\frac{1}{27}(\alpha^3-9\alpha)^2}{(\alpha^2-1)^2};$$

hence Klein's equation

$$\mathbf{J}-1=\frac{(\tau^4+14\tau^3+63\tau^2+70\tau-7)^2}{1728\tau}$$

becomes
$$\frac{\alpha^3-9\alpha}{\alpha^2-1}=\frac{p^8+14p^6+63p^4+70p^2-7}{8p};$$

or say
$$p^8+14p^6+63p^4+70p^2-8\left(\frac{\alpha^3-9\alpha}{\alpha^2-1}\right)p-7=0,$$

(which is the equation p. 148 with p for z), viz. this is the $p\alpha$-equation connecting α with the new multiplier p. It will be observed that it is of the degree 8 in p, and the degree 3 in α, viz. it resembles herein the foregoing $\rho\alpha$-equation, but the form is very much more simple, inasmuch as the α enters into a single coefficient only. The equation may also be written

$$(p^4+5p^2+1)^3(p^4+13p^2+49)-64\frac{(\alpha^2+3)^3}{(\alpha^2-1)^2}p^2=0.$$

74. Using for shortness a single letter m to denote the value $i\sqrt{3}$, we have

$$\frac{\alpha^3-9\alpha+3m(\alpha^2-1)}{\alpha^3-9\alpha-3m(\alpha^2-1)}=\frac{p^8+14p^6+63p^4+70p^2+24mp-7}{p^8+14p^6+63p^4+70p^2-24mp-7};$$

that is
$$\left(\frac{\alpha+m}{\alpha-m}\right)^3=\frac{(p^2-mp+1)^3(p^2+3mp-7)}{(p^2+mp+1)^3(p^2-3mp-7)},$$

or say
$$\frac{\alpha+m}{\alpha-m}=\frac{p^2-mp+1}{p^2+mp+1}\sqrt[3]{\frac{p^2+3mp-7}{p^2-3mp-7}},$$

which is another form of the $p\alpha$-equation.

75. We had $\tau=p^2$, and similarly writing $\tau'=q^2$, then $\tau\tau'=49=p^2q^2$; it must be assumed that $pq=-7$; β is then the same function of q which α is of p, viz. we have
$$\frac{\beta+m}{\beta-m}=\frac{q^2-mq+1}{q^2+mq+1}\sqrt[3]{\frac{q^2+3mq-7}{q^3-3mq-7}};$$

These equations in α and β contain the same cubic radical, viz. we have

$$q^2+3mq-7,=\frac{49}{p^2}-\frac{21m}{p}-7,=-\frac{7}{p^2}(p^2+3mp-7),$$

and similarly

$$q^2-3mq-7 \qquad\qquad =-\frac{7}{p^2}(p^2-3mp-7).$$

Moreover

$$q^2-\ \ mq+1,=\frac{49}{p^2}+\frac{7m}{p}+1,=\frac{1}{p^2}(p^2+7mp+49),$$

and similarly

$$q^2+\ \ mq+1 \qquad\qquad =\frac{1}{p^2}(p^2-7mp+49),$$

and we thus obtain

$$\frac{\beta+m}{\beta-m}=\frac{p^2+7mp+49}{p^2-7mp+49}\sqrt[3]{\frac{p^2+3mp-7}{p^2-3mp-7}},$$

whence, eliminating the cubic radical,

$$\frac{\beta+m}{\beta-m}=\frac{p^2+7mp+49}{p^2-7mp+49}\ \frac{p^2+mp+1}{p^2-mp+1}\ \frac{\alpha+m}{\alpha-m},$$

viz. this gives β as a rational function of α, p. We in fact have

$$\beta=\frac{\alpha(p^4+29p^2+49)-24p(p^2+7)}{\alpha.8p(p^2+7)+(p^4+29p^2+49)}.$$

76. The differential relation $\frac{d\beta}{\beta^2-1}=\frac{\rho^2}{7}\ \frac{d\alpha}{\alpha^2-1}$, substituting therein for ρ its value, becomes

$$\frac{d\beta}{(\beta^2-1)^{\frac{2}{3}}}=\frac{p^2}{7}\ \frac{d\alpha}{(\alpha^2-1)^{\frac{2}{3}}}.$$

But, from the expression for $\frac{\alpha+m}{\alpha-m}$, we obtain

$$d\alpha\left(\frac{1}{\alpha+m}-\frac{1}{\alpha-m}\right)$$

$$=dp\left\{\left(\frac{2p-m}{p^2-mp+1}-\frac{2p+m}{p^2+mp+1}\right)+\frac{1}{3}\left(\frac{2p+3m}{p^2+3mp-7}-\frac{2p-3m}{p^2-3mp-7}\right)\right\},$$

or, omitting from each side a factor $-2m$,

$$\frac{d\alpha}{\alpha^2+3}=dp\left(\frac{-p^2+1}{p^4+5p^2+1}+\frac{p^2+7}{p^4+13p^2+49}\right)=\frac{56dp}{(p^4+5p^2+1)(p^4+13p^2+49)}.$$

But we have, No. 73,

$$\frac{\alpha^2+3}{(\alpha^2-1)^{\frac{2}{3}}} \qquad\qquad =\frac{(p^4+5p^2+1)(p^4+13p^2+49)^{\frac{1}{3}}}{4p^{\frac{2}{3}}},$$

and thence

$$\frac{d\alpha}{(\alpha^2-1)^{\frac{2}{3}}}=\frac{14dp}{p^{\frac{2}{3}}(p^4+13p^2+49)^{\frac{2}{3}}},$$

and similarly

$$\frac{d\beta}{(\beta^2-1)^{\frac{2}{3}}}=\frac{14dq}{q^{\frac{2}{3}}(q^4+13q^2+49)^{\frac{2}{3}}}.$$

The equation $q=-\frac{7}{p}$ gives

$$dq=\frac{7dp}{p^2},\ q^{\frac{2}{3}}(q^4+13q^2+49)^{\frac{2}{3}}=49p^{-\frac{10}{3}}(p^4+13p^2+49)^{\frac{2}{3}},$$

and we thence have

$$\frac{d\beta}{(\beta^2-1)^{\frac{2}{3}}}=\frac{2p^{\frac{4}{3}}dp}{(p^4+13p^2+49)^{\frac{2}{3}}},=\frac{p^2}{7}\,\frac{d\alpha}{(\alpha^2-1)^{\frac{2}{3}}},$$

the required relation.

77. From the value of ρ we have

$$\frac{d\rho}{\rho}=\frac{dp}{p}+\frac{\frac{1}{3}\alpha d\alpha}{\alpha^2-1}-\frac{\frac{1}{3}\beta d\beta}{\beta^2-1},$$

which, substituting for $d\beta$ its value, becomes

$$=\frac{dp}{p}+\frac{\frac{1}{3}d\alpha}{(\alpha^2-1)^{\frac{2}{3}}}\left\{\frac{\alpha}{(\alpha^2-1)^{\frac{1}{3}}}-\frac{\beta}{(\beta^2-1)^{\frac{1}{3}}}\,\frac{p^2}{7}\right\},$$

or say

$$\frac{1}{\rho}\,\frac{d\rho}{d\alpha}=\frac{1}{p}\,\frac{dp}{d\alpha}+\frac{\frac{1}{3}}{(\alpha^2-1)^{\frac{2}{3}}}\left\{\frac{\alpha}{(\alpha^2-1)^{\frac{1}{3}}}-\frac{\beta}{(\beta^2-1)^{\frac{1}{3}}}\,\frac{p^2}{7}\right\},$$

which, however, is more conveniently written

$$\frac{1}{\rho}\,\frac{d\rho}{d\alpha}=\frac{1}{p}\,\frac{dp}{d\alpha}+\frac{\frac{1}{3}}{\alpha^2-1}(\alpha-\beta\rho^2);$$

and then substituting in the formulæ for A_1, A_2 we find

$$A_1=7(\alpha^2-1)\frac{1}{p}\,\frac{dp}{d\alpha}-\frac{7}{6}\alpha+\frac{1}{6}\beta\rho^2;$$

$$\frac{1}{\rho}A_2=7(\alpha^2-1)\frac{1}{p}\,\frac{dp}{d\alpha}-\frac{5}{6}\alpha-\frac{1}{6}\beta\rho^2,$$

(expressions which give, as they should do, $A_2-\rho A_1=\frac{1}{3}(\alpha\rho-\beta\rho^3)$). In these last formulæ ρ is to be regarded as standing for its value, $=p\,\frac{\sqrt[6]{\alpha^2-1}}{\sqrt[6]{\beta^2-1}}$.

78. To further reduce these values, consider the expression of β given No. 75. If for a moment we represent this by

$$\beta = \frac{F\alpha - 3G}{G\alpha + F}, \text{ where } F = p^4 + 29p^2 + 49,\ G = 8p(p^2 + 7),$$

then we have

$$\beta^2 - 1 = \frac{(F^2 - G^2)\alpha^2 - 8FG\alpha + 9G^2 - F^2}{(G\alpha + F)^2},$$

or, multiplying the numerator and denominator each by $G\alpha + F$, so as to make the denominator a perfect cube, the numerator becomes

$$G(F^2 - G^2)(\alpha^3 - 9\alpha) + F(F^2 - 9G^2)(\alpha^2 - 1),$$

and putting for the factor G of the first term its value $= 8p(p^2 + 7)$, we thus obtain

$$\frac{\beta^2 - 1}{\alpha^2 - 1} = \frac{(F^2 - G^2)(p^2 + 7)\,8p\left(\dfrac{\alpha^3 - 9\alpha}{\alpha^2 - 1}\right) + F(F^2 - 9G^2)}{(G\alpha + F)^3},$$

viz. in virtue of the $p\alpha$-equation, this is

$$\frac{\beta^2 - 1}{\alpha^2 - 1} = \frac{(F^2 - G^2)(p^2 + 7)(p^8 + 14p^6 + 63p^4 + 70p^2 - 7) + F(F^2 - 9G^2)}{(G\alpha + F)^3}.$$

This numerator is $= (p^4 + 5p^2 + 1)^3 p^6$; in fact we have

$$\begin{aligned} (F^2 - G^2)(p^2 + 7) &= p^{10} + p^8 + p^6 + 7p^4 + 343p^2 + 16807, \\ F^2 - 9G^2 &= p^8 - 518p^6 - 7125p^4 - 25382p^2 + 2401, \end{aligned}$$

and thence forming the two terms of the numerator and adding them together—for shortness I write down only the coefficients—we have

$$\begin{array}{rrrrrrrrrr} 1 & 15 & 78 & 154 & 567 & 22113 & 257390 & 1082802 & 1174089 & -117649 \\ & & & 1 & -489 & -22098 & -257389 & -1082802 & -1174089 & 117649 \\ \hline =1 & 15 & 78 & 155 & 78 & 15 & 1 & 0 & 0 & 0 \end{array}$$

viz. these are the coefficients of $(p^4 + 5p^2 + 1)^3 p^6$. Hence

$$\frac{\beta^2 - 1}{\alpha^2 - 1} = \frac{(p^4 + 5p^2 + 1)^3 p^6}{(G\alpha + F)^3};$$

or, extracting the cube root, and for G, F substituting their values,

$$\frac{\sqrt[3]{\beta^2 - 1}}{\sqrt[3]{\alpha^2 - 1}} = \frac{(p^4 + 5p^2 + 1)p^2}{8p(p^2 + 7)\alpha + p^4 + 29p^2 + 49},$$

and thence also

$$\rho^2 = \frac{8p(p^2 + 7)\alpha + p^4 + 29p^2 + 49}{p^4 + 5p^2 + 1},$$

viz. we have thus ρ^2 expressed as a rational function of p, α.

79. It will presently appear that ρ is in fact expressible as a rational function of p, α, but I am unable to obtain this expression in a simple form. Admitting that ρ is thus expressible, a direct process for obtaining the expression is as follows. Writing

$$\xi = \frac{8p(p^2+7)\alpha + p^4 + 29p^2 + 49}{p^4 + 5p^2 + 1} \quad (=\rho^2),$$

and by means hereof introducing ξ in place of α into the equation

$$p^8 + 14p^6 + 63p^4 + 70p^2 - 8p\,\frac{\alpha^3 - 9\alpha}{\alpha^2 - 1} - 7 = 0,$$

we have for ξ a cubic equation,

$$a\xi^3 + b\xi^2 + c\xi + d = 0,$$

where the coefficients a, b, c, d are given rational functions of p. This equation may be written

$$a\xi(\xi + \vartheta)^2 + b'\xi^2 + c'\xi + d = 0,$$

where $b' = b - 2a\vartheta$, $c' = c - a\vartheta^2$; and the last three terms will be a square if only $c'^2 - 4b'd = 0$; that is, if

$$(a\vartheta^2 - c)^2 + 4d(2a\vartheta - b) = 0,$$

a biquadratic equation in ϑ which (ρ being expressible as above) must have one of its roots $=$ a rational function of p. Calling this ϑ, we then have

$$a\xi(\xi + \vartheta)^2 + \frac{1}{b'}\left(b'\xi + \tfrac{1}{2}c'\right)^2 = 0, \text{ or say } a\rho^2(\xi + \vartheta)^2 + \frac{1}{b'}\left(b'\xi + \tfrac{1}{2}c'\right)^2 = 0,$$

hence

$$\rho = \sqrt{\frac{-1}{ab'}}\cdot\frac{b'\xi + \frac{1}{2}c'}{\xi + \vartheta},$$

where ξ denotes a linear function of α as above; the quadric radical will have a rational value, and the form of the equation thus is

$$\rho = \frac{A\alpha + B}{C\alpha + D},$$

where A, B, C, D are rational and integral functions of p. But I am not able to carry out the process.

80. As shown, No. 78, we have

$$\rho^2 = \frac{8p(p^2+7)\alpha + p^4 + 29p^2 + 49}{p^4 + 5p^2 + 1}.$$

Multiplying by the value of β, *ante* No. 75, we find

$$\beta\rho^2 = \frac{(p^4 + 29p^2 + 49)\alpha - 24p(p^2+7)}{p^4 + 5p^2 + 1}.$$

and we can hence find A_1 and A_2 by the formulæ

$$A_1 = 7(\alpha^2 - 1)\frac{1}{p}\frac{dp}{d\alpha} - \frac{7}{6}\alpha + \frac{1}{6}\beta\rho^2;$$

$$\frac{1}{\rho}A_2 = 7(\alpha^2 - 1)\frac{1}{p}\frac{dp}{d\alpha} - \frac{5}{6}\alpha - \frac{1}{6}\beta\rho^2,$$

or, for the second of these we may write

$$\frac{1}{\rho}A_2 = A_1 + \frac{1}{3}(\alpha - \beta\rho^2).$$

But in a different point of view, regarding only ρ^2, but not ρ, as a given function of p, α, we must to these equations join the equation $12A_2 = 6A_1^2 + 8\alpha A_1 - \rho^4 + 7$, *ante* No. 69, and we have thus equations for the determination of A_1, A_2, and ρ.

81. We have

$$A_1 = \frac{(p^4+5p^2+1)(p^4+13p^2+49)}{8p}\,\frac{\alpha^2-1}{\alpha^2+3}$$
$$\qquad - \frac{7}{6}\alpha + \frac{\alpha(p^4+29p^2+49) - 24p(p^2+7)}{6(p^4+5p^2+1)},$$

where the second line is

$$= \frac{\alpha(-p^4-p^2+7) - 4p(p^2+7)}{p^4+5p^2+1}.$$

Uniting the two terms, we have a denominator $8p(p^4+5p^2+1)$, and in the numerator a term $8p\alpha^3$ which may be got rid of by means of the $p\alpha$-equation; the numerator thus becomes

$$\begin{aligned} = {} & 96p(-p^4-p^2+7) - 128p^2(p^2+7)\alpha \\ & + (\alpha^2-1)\{(-p^4-p^2+7)(p^8+14p^6+63p^4+70p^2-7)\} \\ & + (p^4+5p^2+1)^2(p^4+13p^2+49) - 32p^2(p^2+7), \end{aligned}$$

where the whole divides by $8p$; and we finally obtain

$$A_1 = \frac{12(-p^4-p^2+7) - 16p(p^2+7)\alpha + (\alpha^2-1)p(p^8+17p^6+102p^4+225p^2+97)}{(\alpha^2+3)(p^4+5p^2+1)}.$$

Proceeding to calculate the value of $A_1 + \frac{1}{3}(\alpha - \beta\rho^2)$, we then have

$$\frac{1}{3}(\alpha - \beta\rho^2) = \frac{-8(p^2+2)\alpha + 8p(p^2+7)}{p^4+5p^2+1}.$$

Multiplying the numerator and denominator by $\alpha^2 + 3$, we have in the numerator a term in $8\alpha^3$ which may be got rid of by means of the $p\alpha$-equation; the numerator thus becomes

$$12(-p^4-9p^2-9)+16p(p^2+7)+(\alpha^2-1)p\{p^8+17p^6+102p^4+225p^2+97\}$$
$$-\frac{p^2+2}{p^2}(p^8+14p^6+63p^4+70p^2-7)+8(p^2+7),$$

and we finally obtain

$$\frac{1}{\rho}A_2=\frac{12(-p^4-9p^2-9)+16p(p^2+7)+(\alpha^2-1)p^{-1}(p^8+11p^6+37p^4+20p^2+2)}{(\alpha^2+3)(p^4+5p^2+1)}.$$

82. The expressions obtained above for ρ^2, A_1, A_2 are of the form

$$\rho^2=\frac{M+N\alpha}{S},\ A_1=\frac{P_1+Q_1\alpha+R_1\alpha^2}{S(\alpha^2+3)},\ \frac{1}{\rho}A_2=\frac{P_2+Q_2\alpha+R_2\alpha^2}{S(\alpha^2+3)},$$

where

$$\begin{aligned}
&M=p^4+29p^2+49; && N=8p(p^2+7); && S=p^4+5p^2+1,\\
&P_1=12(-p^4-p^2+7)-p(p^8+17p^6+102p^4+225p^2+97), && && Q_1=-16p(p^2+7),\\
&R_1= && p(p^8+17p^6+102p^4+225p^2+97);\\
&P_2=12(-p^4-9p^2-9)-p^{-1}(p^8+11p^6+37p^4+20p^2+2), && && Q_2=16p(p^2+7),\\
&R_2= && p^{-1}(p^8+11p^6+37p^4+20p^2+2);
\end{aligned}$$

and substituting these values in the foregoing equation

$$12A_2=6A_1^2+8\alpha A_1-\rho^4+7,$$

we obtain

$$12\rho\left\{\frac{P_2+Q_2\alpha+R_2\alpha^2}{S(\alpha^2+3)}\right\}=\left\{\frac{6(P_1+Q_1\alpha+R_1\alpha^2)^2}{S^2(\alpha^2+3)^2}+8\alpha\frac{P_1+Q_1\alpha+R_1\alpha^2}{S(\alpha^2+3)}-\frac{(M+N\alpha)^2}{S^2}+7\right\};$$

that is,

$$\rho=\frac{1}{12(P_2+Q_2\alpha+R_2\alpha^2)^2(3+\alpha^2)S}\{6(P_1+Q_1\alpha+R_1\alpha^2)^2+8\alpha S(3+\alpha^2)(P_1+Q_1\alpha+R_1\alpha^2)$$
$$-(M+N\alpha)^2(3+\alpha^2)^2+7S^2(3+\alpha^2)^2\},$$

which, by means of the $p\alpha$-equation

$$p^8+14p^6+63p^4+70p^2-\left(\frac{\alpha^3-9\alpha}{\alpha^2-1}\right)8p-7=0,$$

should be reducible to the form

$$\rho=A\alpha^2+B\alpha+C,\ \text{or}\ \rho=\frac{A\alpha+B}{C\alpha+D};$$

but I have not been able to obtain in either of these forms a simple expression of ρ as a function of p, α. Supposing it obtained, the $\rho\alpha$-equation, *ante* No. 51, would of course be thereby transformable into the foregoing $p\alpha$-equation. And considering p as an auxiliary parameter thus introduced into the formulæ in place of ρ, then β and the coefficients A_1, A_2 are, by what precedes, expressed in terms of p, α; that is, in effect in terms of ρ, α, and we thus have the formulæ of transformation for the $\rho\alpha\beta$-form.

83. There exists a remarkably simple particular case. Write for convenience $\theta = \sqrt{7}$; the $p\alpha$-equation is satisfied by the values $p = -\theta$, $\alpha = -\frac{3}{8}\theta$. In fact, these values give $8p\alpha = 3\theta^2, = 21, \frac{\alpha^2 - 9}{\alpha^2 - 1} = \left(\frac{63}{64} - 9\right) \div \left(\frac{63}{64} - 1\right), = 513$; the term in α is thus $21.513, = 10773$; but, assuming $p^2 = 7$, we have $p^8 + 14p^6 + 63p^4 + 70p^2 - 7 = 2401 + 4802 + 3087 + 490 - 7; = 10773$, and the equation is thus satisfied. And these values, $p = -\theta$, $\alpha = -\frac{3}{8}\theta$, give $\rho^2 = 7$, $\beta = \frac{3}{8}\theta$, $A_1 = 2\theta$, $A_2 = \rho\theta$; the equation $12A_2 = 6A_1^2 + 8\alpha A_1 - \rho^4 + 7$ thus becomes $12\rho\theta = 168 - 42 - 49 + 7, = 84$; that is, $\rho\theta = 7, = \theta^2$, or $\rho = \theta\ (= -p)$. We have $\alpha^2 - 1, = \beta^2 - 1, = -\frac{1}{64}$; but from the equation $\rho = p\frac{\sqrt[6]{\alpha^2 - 1}}{\sqrt[6]{\beta^2 - 1}}$, it appears that the sixth roots must be equal with opposite signs, say $\sqrt[6]{\alpha^2 - 1} = \frac{i}{2}$, $\sqrt[6]{\beta^2 - 1} = \frac{-i}{2}$. Retaining θ to stand for its value $= \sqrt{7}$, the differential equation is

$$\frac{dy}{\sqrt{1 - \frac{3}{4}\theta y^2 + y^4}} = \frac{\theta dx}{\sqrt{1 + \frac{3}{4}\theta x^2 + x^4}},$$

satisfied by

$$y = \frac{x(\theta + 7x^2 + 2\theta x^4 + x^6)}{1 + 2\theta x^2 + 7x^4 + \theta x^6}.$$

It may be remarked that the quartic functions of y and x resolved into their linear factors are

$$\left\{y + \frac{3i + \theta}{2\sqrt{2}(1 + i)}\right\}\left\{y + \frac{3i - \theta}{2\sqrt{2}(1 + i)}\right\}\left\{y + \frac{-3i + \theta}{2\sqrt{2}(1 - i)}\right\}\left\{y + \frac{-3i - \theta}{2\sqrt{2}(1 - i)}\right\}$$

and

$$\left\{x + \frac{3 - i\theta}{2\sqrt{2}(1 + i)}\right\}\left\{x + \frac{3 + i\theta}{2\sqrt{2}(1 + i)}\right\}\left\{x + \frac{3 - i\theta}{2\sqrt{2}(1 - i)}\right\}\left\{x + \frac{3 + i\theta}{2\sqrt{2}(1 - i)}\right\},$$

and that for the first of the y-factors, substituting for y its value, we have

$$x^7 + 2\theta x^5 + 7x^3 + \theta x + \frac{3i + \theta}{2\sqrt{2}(1 + i)}(\theta x^6 + 7x^4 + 2\theta x^2 + 1)$$
$$= \left(x + \frac{3 - i\theta}{2\sqrt{2}(1 + i)}\right)\left\{x^3 + \frac{1 + i\theta}{\sqrt{2}(1 + i)}x^2 + \frac{1}{2}(i + \theta)x + \frac{1 + i}{\sqrt{2}}\right\}^2,$$

with like expressions for the other y-factors respectively.

Brioschi's Transformation Theory. Art. No. 84.

84. M. Brioschi has kindly referred me to two papers by him, "Sur une Formule de Transformation des Fonctions Elliptiques," Comptes Rendus, t. 79 (1874), pp. 1065–1069, and ibid. t. 80 (1875), pp. 261–264. They relate to the form

$$\frac{dx}{\sqrt{4x^3-g_2x-g_3}}=\frac{dy}{\sqrt{4y^3-G_2y-G_3}},$$

with a formula of transformation

$$y=\frac{U}{T^2},\ T=x^\nu+a_1x^{\nu-1}+a_2x^{\nu-2}\dots+a_\nu\ \left\{\nu=\frac{1}{2}(n-1)\right\}$$
$$U=x^n+\alpha_1x^{n-2}+\alpha_2x^{n-3}\dots+\alpha_\nu.$$

The general theory for any value of n is developed to a considerable extent, and it would without doubt give very interesting results for the case $n=7$; but the formulæ are only completely worked out for the preceding two cases $n=3$ and $n=5$. For these cases the formulæ are as follows:

Cubic transformation: $n=3$,

$$y=\frac{x^3+\alpha_1x^2+\alpha_2x+\alpha_3}{(x+a_1)^2}.$$

Corresponding to the modular equation we have

$$a_1^4-\frac{1}{2}g_2a_1^2+g_3a_1-\frac{1}{48}g_2^2=0,$$

and then

$$G_2-9g_2=6(20a_1^2-3g_2),\quad G_3+27g_3=-14(20a_1^2-3g_2)a_1,$$

whence also

$$a_1=-\frac{3}{7}\frac{G_3+27g_3}{G_2-9g_2},$$

and by the general theory α_1, α_2, α_3 are given rationally in terms of a_1, g_2, g_3.

Quintic transformation: $n=5$,

$$y=\frac{x^5+\alpha_1x^4+\alpha_2x^3+\alpha_3x^2+\alpha_4x+\alpha_5}{(x^2+a_1x+a_2)^2}.$$

We have

$$a_1X-2Y=0,\quad (12a_1^2+g_2)X-30a_1Y=0,$$

where

$$X=a_1^3-6a_1^2a_2+\frac{1}{2}g_2a_1-g_3,$$

$$Y=5a_2^2-a_1^2a_2+\frac{1}{2}g_2a_2-g_3a_1+\frac{1}{16}g_2^2.$$

The first of these gives

$$a_2 = \frac{1}{6a_1}\left(a_1^3 + \frac{1}{2} g_2 a_1 - g_3\right),$$

and then eliminating a_2, we have, corresponding to the modular equation,

$$a_1^6 - 5g_2a_1^4 + 40g_3a_1^3 - 5g_2^2a_1^2 + 8g_2g_3a_1 - 5g_3^2 = 0.$$

We then have

$$G_2 - 25g_2 = \frac{8}{a_1}(10a_1^3 - 8g_2a_1 + 5g_3), \quad G_3 + 125g_3 = -14(10a_1^3 - 8g_2a_1 + 5g_3);$$

whence also

$$a_1 = -\frac{4}{7}\frac{G_3 + 125g_3}{G_2 - 25g_2},$$

and by the general theory $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ are given rationally in terms of a_1, g_2, g_3.

These results are contained in the former of the papers above referred to; the latter contains some properties of these modular equations.

Symbolic Treatment of Exact Linear Differential Equations.

By Wm. Woolsey Johnson.

1. The linear equation is here supposed to consist of terms of the form

$$Ax^s\frac{d^ry}{dx^r}, \text{ or } Ax^sD^ry,$$

in which r is zero or a positive integer, and s is unrestricted. Let these terms be grouped together in such a way that the value of $s-r$ is the same for all the terms in a group; then, if m be the least value of r for the terms in a group, and $q-m=s-r$, the group may be written

$$x^q\left[A_0+A_1xD+A_2x^2D^2+\dots\right]D^my. \tag{1}$$

Using ϑ to denote the operator $x\dfrac{d}{dx}$ or xD, we may reduce this expression by means of the theorem

$$x^nD^n=\vartheta(\vartheta-1)(\vartheta-2)\dots(\vartheta-n+1), \tag{A}$$

to the form

$$x^qf(\vartheta)D^my. \tag{2}$$

2. In order that the differential equation may be exact, each group of the form (2) contained in it must separately constitute an exact derivative; and the process of direct integration is equivalent to that of resolving, in the case of each group, the symbolic operator into factors of which that most remote from the operand is the simple factor D. It is well known that if m is not zero and q is an integer less than m, every term in (1), and therefore the group itself, is an exact derivative. The symbolic transformation of the expression (2) may in this case be effected by the formula deduced below.

3. We have by differentiation

$$D.xDy=xD^2y+Dy,$$

or

$$D\vartheta y=\vartheta Dy+Dy;$$

whence, symbolically,

$$\vartheta D=D(\vartheta-1). \tag{3}$$

Operating upon both members of this equation with ϑ, we have

$$\vartheta^2 D = \vartheta D(\vartheta - 1) = D(\vartheta - 1)^2,$$

and in like manner we find generally

$$\vartheta^r D = D(\vartheta - 1)^r;$$

whence, f denoting a rational integral function,

$$f(\vartheta)D = Df(\vartheta - 1). \tag{4}$$

Again, operating with each member of this equation upon Dy, we have

$$f(\vartheta)D^2 = Df(\vartheta - 1)D = D^2 f(\vartheta - 2),$$

and in like manner,

$$f(\vartheta)D^3 = D^2 f(\vartheta - 2)D = D^3 f(\vartheta - 3),$$

and generally,

$$f(\vartheta)D^m = D^m f(\vartheta - m). \tag{B}$$

4. Applying this formula, in the case mentioned above, in which $m - q$ is a positive integer, to the expression (2), we have

$$x^q f(\vartheta)D^m = x^q D^m f(\vartheta - m) = x^q D^q D^{m-q} f(\vartheta - m).$$

By the formula (A) this becomes

$$x^q f(\vartheta)D^m = \vartheta(\vartheta - 1)\ldots.(\vartheta - q + 1)D^{m-q} f(\vartheta - m),$$

and, making a second application of formula (B), we have

$$x^q f(\vartheta)D^m = D^{m-q}(\vartheta - m + q)\ldots.(\vartheta - m + 1)f(\vartheta - m), \tag{5}$$

in which a resolution of the operator into symbolic factors of the form required is effected.

5. Formula (A), by which we reduce the groups to the form (2), is also readily deduced from equation (3); for, multiplying the latter by x, we have

$$x^2 D^2 = \vartheta(\vartheta - 1);$$

and operating with each member of this equation upon Dy, we find

$$x^2 D^3 = \vartheta(\vartheta - 1)D,$$

which, by equation (4), becomes

$$x^2 D^2 = D(\vartheta - 1)(\vartheta - 2),$$

and multiplying by x,

$$x^3 D^3 = \vartheta(\vartheta - 1)(\vartheta - 2).$$

Thus we may successively derive all the results included in equation (A).

6. When m is zero, and also when m is an integer, unless q is zero or a positive integer less than m, the possibility of resolving the operator into factors of the required form depends upon the existence of a proper factor in $f(\vartheta)$.

By differentiation we have

$$Dx^{q+1}y = x^{q+1}Dy + (q + 1)x^q y;$$

whence, using Dx^{q+1} as a symbol of operation,

$$x^q(\vartheta + q + 1) = Dx^{q+1}. \tag{C}$$

Hence, if $\vartheta + q + 1$ is a factor of $f(\vartheta)$, that is, if $-(q+1)$ is a root of $f(\vartheta) = 0$, so that

$$f(\vartheta) = (\vartheta + q + 1)\phi(\vartheta), \tag{6}$$

ϕ being rational and integral, then the expression (2) may be transformed thus,

$$x^q f(\vartheta) D^m y = Dx^{q+1}\phi(\vartheta) D^m y, \tag{7}$$

in which, subject to the condition (6), the operator is resolved into factors of the required form.

7. This condition includes that mentioned in (2); for, if we transform the operator by formulae (B) and (A) thus,

$$\begin{aligned} x^q f(\vartheta) D^m &= x^{q-m} x^m D^m f(\vartheta - m) \\ &= x^{q-m}\vartheta(\vartheta - 1) \ldots . (\vartheta - m + 1) f(\vartheta - m), \end{aligned} \tag{8}$$

the condition of direct integrability is the occurrence of the factor $\vartheta + q - m + 1$. Now if q is an integer less than m, this factor occurs among those actually written in equation (8); thus the condition is in that case satisfied independently of the form of f. Otherwise, the factor can only occur as a factor of $f(\vartheta - m)$, and the condition of direct integrability then is that $f(\vartheta - m)$ shall vanish with $\vartheta + q - m + 1$; that is, that $f(-q-1) = 0$, as before. Thus nothing is gained by the transformation (8), and it is better to employ the transformation (5) when, m being a positive integer, q is zero or an integer less than m, and the transformation (7), when, this condition not being fulfilled, $f(\vartheta)$ satisfies the condition (6).

8. The criteria thus established show at once by what powers of x the group being multiplied becomes exact, and thus, in an equation containing two or more groups, whether there be an integrating factor of the form x^p. For example, the equation

$$2x^2(x+1)\frac{d^2y}{dx^2} + x(7x+3)\frac{dy}{dx} - 3y = X$$

contains two groups, which being written in the form (2), the equation is

$$x^2(2\vartheta + 7)Dy + (2\vartheta + 3)(\vartheta - 1)y = X.$$

The first term becomes exact according to the first condition when multiplied by x^{-2}, and exact according to the second condition when multiplied by $x^{\frac{1}{2}}$. The second term becomes exact in this case when multiplied by either of the same factors. Hence we may write the given operator

$$2x^2(x+1)D^2 + x(7x+3)D - 3$$

in either of the forms

$$x^2 D\left[(2\vartheta + 5) + x^{-1}(2\vartheta + 3)\right],$$

or

$$2x^{-\frac{1}{2}} D\left[x^{\frac{7}{2}} D + x^{\frac{3}{2}}(\vartheta - 1)\right].$$

Again, in the first of these expressions, each of the two terms between the brackets fulfills the second condition when multiplied by $x^{\frac{3}{2}}$; the equation may therefore be further reduced to

$$2x^2 D x^{-\frac{3}{2}} D\left(x^{\frac{5}{2}} + x^{\frac{3}{2}}\right) y = X.$$

The value of y obtained by performing on X the inverse operations in their proper order is

$$y = \frac{1}{2x^{\frac{3}{2}}(x+1)} \int x^{\frac{3}{2}} \int x^{-2} X dx dx.$$

In like manner the equation might be reduced to

$$2x^{-\frac{1}{2}} D x^{\frac{7}{2}} D\left(1 + x^{-1}\right) y = X,$$

giving

$$y = \frac{x}{2(x+1)} \int x^{-\frac{7}{2}} \int x^{\frac{1}{2}} X dx dx;$$

but perhaps the best expression for y is that which results from elimination between the two first integrals, namely,

$$y = \frac{x}{5(x+1)} \int x^{-2} X dx - \frac{1}{5x^{\frac{3}{2}}(x+1)} \int x^{\frac{1}{2}} X dx.$$

It is noticeable that, whenever an equation of the form considered is susceptible of two successive direct integrations with intermediate multiplication by a power of x, it is also susceptible of direct integration when multiplied by either of two different powers of x.

9. As a second example, let us take the equation

$$(x^3 - x)\frac{d^3y}{dx^3} + (8x^2 - 3)\frac{d^2y}{dx^2} + 14x\frac{dy}{dx} + 4y = 0,$$

which, when reduced by formula (A), becomes

$$(\vartheta + 1)(\vartheta + 2)^2 y - (\vartheta + 3) D^2 y = 0.$$

Reducing the first term by a double application of formula (C), and the second by formula (B), the equation becomes

$$D^2\left[(x^3 - x) D + 2x^2 - 1\right] y = 0.$$

In this case there is no intermediate x-factor between the D's, and the symbolic operator

$$(x^3 - x) D^3 + (8x^2 - 3) D^2 + 14xD + 4$$

does not admit of resolution into factors of the first order in two separate ways.

10. As a final example, consider the equation

$$(2x^2 + x^{\frac{3}{2}})\frac{d^3y}{dx^3} + 3x\frac{d^2y}{dx^2} - 3\frac{dy}{dx} = X,$$

which becomes $$(2\vartheta + 3)(\vartheta - 1)Dy + x^{\frac{1}{2}}\vartheta D^2y = X.$$

The first term becomes exact by the second condition when multiplied by $x^{\frac{1}{2}}$, and the second term will then be exact by the first condition. The first member of the equation thus becomes

$$x^{-\frac{1}{2}}D\left[2x^{\frac{3}{2}}(\vartheta - 1)D + (\vartheta - 1)(\vartheta - 2)\right]y,$$

in which $\vartheta - 1$ is not a factor since it is not commutative with D; but, transforming the first term by formula (B), the expression becomes

$$x^{-\frac{1}{2}}D\left[(2x^{\frac{3}{2}} + x)D - 1\right](xD - 2)y,$$

in which the operator is resolved into three factors of the first order.

In the original expression we might, however, have reserved the factor D which stands nearest to the operand, thus,

$$\left[(2\vartheta + 3)(\vartheta - 1) + x^{\frac{1}{2}}\vartheta D\right]Dy = X.$$

The factor in parenthesis cannot now be made exact; but, transforming the second term by formula (B), the first member may be written

$$\left[(2x + x^{\frac{1}{2}})D + 3\right](xD - 1)Dy,$$

in which a second resolution of the operator into factors of the first order has been effected.

Solvable Quintic Equations with Commensurable Coefficients.

By George Paxton Young, *University College, Toronto, Canada.*

Object of the Paper.

§1. Some time ago, in the *American Journal of Mathematics* (Vol. VI, page 103), the present writer sketched a general method for finding the roots of solvable irreducible equations of the fifth degree. The method was partially developed, and its application to certain forms of quintic equations was shown. It is now proposed to give the method the farther development necessary to make it applicable, by a definite and certain process, and without any difficulty beyond the labor of operation, to all solvable irreducible quintics having commensurable coefficients. The following equations will be solved as examples of the application of the theory:

1. $x^5 + 3x^2 + 2x - 1 = 0.$
2. $x^5 - 10x^3 - 20x^2 - 1505x - 7412 = 0.$
3. $x^5 + \frac{625}{4}x + 3750 = 0.$
4. $x^5 - \frac{22}{5}x^3 - \frac{11}{25}x^2 + \frac{11 \times 42}{125}x + \frac{11 \times 89}{3125} = 0.$
5. $x^5 + 20x^3 + 20x^2 + 30x + 10 = 0.$
6. $x^5 + 320x^2 - 1000x + 4288 = 0.$
7. $\left(\frac{x}{10}\right)^5 + 40\left(\frac{x}{10}\right)^2 - 69\left(\frac{x}{10}\right) + 108 = 0.$
8. $x^5 - 20x^3 + 250x - 400 = 0.$
9. $x^5 - 5x^3 + \frac{85}{8}x - \frac{13}{2} = 0.$
10. $x^5 + \frac{20x}{17} + \frac{21}{17} = 0.$

11. $x^5 - \frac{4x}{13} + \frac{29}{65} = 0.$
12. $x^5 + \frac{10x}{13} + \frac{3}{13} = 0.$
13. $x^5 + 110(5x^3 + 60x^2 + 800x + 8320) = 0.$
14. $x^5 - 20x^3 - 80x^2 - 150x - 656 = 0.$
15. $x^5 - 40x^3 + 160x^2 + 1000x - 5888 = 0.$
16. $\left(\frac{x}{2}\right)^5 - 50\left(\frac{x}{2}\right)^3 - 600\left(\frac{x}{2}\right)^2 - 2000\left(\frac{x}{2}\right) - 11200 = 0.$
17. $x^5 + 110(5x^3 + 20x^2 - 360x + 800) = 0.$
18. $x^5 - 20x^3 + 320x^2 + 540x + 6368 = 0.$
19. $x^5 - 20x^3 - 160x^2 - 420x - 8928 = 0.$
20. $x^5 - 20x^3 + 170x + 208 = 0.$

The first equation in this group was brought under the notice of the writer by a mathematical correspondent; the fourth has been treated by Lagrange; the others were formed by the writer with a view to the full illustration of his theory.

THE METHOD.

§2. In the article of the *Journal* above referred to, certain principles were assumed, as having been previously established, or as being known to mathematicians. It was taken for granted that the root x of the solvable irreducible quintic

$$x^5 + p_2x^3 + p_3x^2 + p_4x + p_5 = 0 \tag{1}$$

is of the form

$$\frac{1}{5}\left(\Delta_1^{\frac{1}{5}} + \Delta_2^{\frac{1}{5}} + \Delta_3^{\frac{1}{5}} + \Delta_4^{\frac{1}{5}}\right),$$

or, putting u_1 for $\frac{1}{5}\Delta_1^{\frac{1}{5}}$, u_2 for $\frac{1}{5}\Delta_2^{\frac{1}{5}}$, u_3 for $\frac{1}{5}\Delta_3^{\frac{1}{5}}$, and u_4 for $\frac{1}{5}\Delta_4^{\frac{1}{5}}$,

$$u_1 + u_2 + u_3 + u_4,$$

where u_1, u_2, u_3 and u_4 are the roots of a quartic equation, which, when irreducible, as it is in the most general case, that includes all the others, is a uni-serial Abelian. The expressions u_1, u_2, u_3 and u_4 are such that

$$\left.\begin{aligned} u_1u_4 &= g + a\sqrt{z} \\ u_2u_3 &= g - a\sqrt{z} \end{aligned}\right\};\tag{2}$$

and

$$\left.\begin{aligned} u_1^2u_3 &= k + c\sqrt{z} + (\theta + \phi\sqrt{z})\sqrt{(hz + h\sqrt{z})} \\ u_4^2u_2 &= k + c\sqrt{z} - (\theta + \phi\sqrt{z})\sqrt{(hz + h\sqrt{z})} \\ u_2^2u_1 &= k - c\sqrt{z} + (\theta - \phi\sqrt{z})\sqrt{(hz - h\sqrt{z})} \\ u_3^2u_4 &= k - c\sqrt{z} - (\theta - \phi\sqrt{z})\sqrt{(hz - h\sqrt{z})} \end{aligned}\right\}\tag{3}$$

where g, k, a, c, h, θ and ϕ are rational; and $z = e^2 + 1$, e being rational. It is readily seen that

$$g = -\frac{p_2}{10}, \text{ and } k = -\frac{p_3}{20}.$$

Because $u_1^5 = \frac{(u_1^2 u_3)^2 (u_2^2 u_1)}{(u_2 u_3)^2}$, it follows from (2) and (3) that

$$\left.\begin{aligned} u_1^5 &= B + B'\sqrt{z} + (B'' + B'''\sqrt{z})\sqrt{(hz + h\sqrt{z})} \\ u_4^5 &= B + B'\sqrt{z} - (B'' + B'''\sqrt{z})\sqrt{(hz + h\sqrt{z})} \\ u_2^5 &= B - B'\sqrt{z} + (B'' - B'''\sqrt{z})\sqrt{(hz - h\sqrt{z})} \\ u_3^5 &= B - B'\sqrt{z} - (B'' - B'''\sqrt{z})\sqrt{(hz - h\sqrt{z})} \end{aligned}\right\} \tag{4}$$

where B, B', B'' and B''' are rational functions of a, c, e, h, θ and ϕ. In like manner, because $u_1^3 u_2 = \frac{(u_1^2 u_3)(u_2^2 u_1)}{u_2 u_3}$, we have from (2) and (3)

$$u_1^3 u_2 = A + A'\sqrt{z} + (A'' + A'''\sqrt{z})\sqrt{(hz + h\sqrt{z})},$$

where A, A', A'' and A''' are rational. The value of A is

$$A = \frac{1}{g^2 - a^2 z}\{g(k^2 - c^2 z) + azhe(\theta^2 - \phi^2 z)\}. \tag{5}$$

From these data, the six equations, involving the six unknown quantities a, c, e, h, θ and ϕ, are (see *Journal of Mathematics* as above) obtained:

$$\left.\begin{aligned} p_4 &= -20A + 5g^2 + 15a^2 z \\ p_5 &= -4B + 40acz \\ B'' &= 1 \\ B''' &= 0. \\ hz(\theta^2 + \phi^2 z + 2\theta\phi) &= k^2 + c^2 z - g(g^2 - a^2 z) \\ h(\theta^2 + \phi^2 z + 2\theta\phi z) &= 2kc - a(g^2 - a^2 z) \end{aligned}\right\} \tag{6}$$

Our business is to obtain u_1^5, u_2^5, u_3^5 and u_4^5 from these equations.

§3. It will be found that $a^2 z$ is the root of an equation $F(y) = 0$, whose coefficients are rational functions of p_2, p_3, p_4 and p_5, and which, when p_2 is zero, is of the sixth degree. Since $a^2 z$ is rational, it follows that, when the coefficients of the given quintic are commensurable, the equation $F(y) = 0$ has a commensurable root. Let this be found. Then $a^2 z$ is known. The formulæ from which the equation $F(y) = 0$ is obtained give us, along with $a^2 z$, the value of $\frac{c}{a}$. The remaining elements necessary for the determination of u_1^5, u_2^5, u_3^5 and u_4^5 may then be obtained from linear equations, without finding

a, c, e, h, θ and ϕ separately. Thus a solution of the given quintic is effected. It will be pointed out how a, c, e, h, θ and ϕ can be found separately, should we desire to obtain their values.

THE PROOF.

Case in which p_2 is zero.

§4. When $p_2 = 0$, the investigation is much simplified. By beginning with this case, and presenting a full description of it, we shall be prepared for giving an exposition, less detailed, but still sufficiently minute to make the theory intelligible, of the case in which p_2 is not assumed to be zero. When p_2 is zero, equations (2) and (5) become

$$\left.\begin{aligned} u_1u_4 &= a\sqrt{z} \\ u_2u_3 &= -a\sqrt{z} \\ \text{and}\qquad A &= -\frac{he(\theta^2-\phi^2 z)}{a} \end{aligned}\right\} \tag{7}$$

Since B is the coefficient of the rational part, B' the coefficient of $\sqrt{z}$, B'' the coefficient of $\sqrt{(hz + h\sqrt{z})}$, and B''' the coefficient of $\sqrt{z}\sqrt{(hz + h\sqrt{z})}$, in the expansion of u_1^5, their values are given by the equations

$$\left.\begin{aligned} a^2zB &= 2k(k^2-c^2z) - a^3cz^2 + 2czhe(\theta^2-\phi^2z) \\ a^2zB' &= 2c(k^2-c^2z) + a^3kz + 2khe(\theta^2-\phi^2z) \\ a^2zB'' &= 2(k^2-c^2z)\theta - \frac{2(k^2+c^2z)(\theta+\phi z)}{e} + \frac{z(\theta+\phi)(4kc+a^3z)}{e} \\ a^2zB''' &= 2(k^2-c^2z)\phi + \frac{2(k^2+c^2z)(\theta+\phi)}{e} - \frac{(\theta+\phi z)(4kc+a^3z)}{e} \end{aligned}\right\} \tag{8}$$

The equations (6), when p_2 is zero, become

$$\left.\begin{aligned} p_4 &= -20A + 15a^2z \\ p_5 &= -4B + 40acz \\ B'' &= 1 \\ B''' &= 0 \\ hz(\theta^2+\phi^2z+2\theta\phi) &= k^2+c^2z \\ h(\theta^2+\phi^2z+2\theta\phi z) &= 2kc+a^3z \end{aligned}\right\} \tag{9}$$

$$\left.\begin{aligned} \text{Let}\qquad y &= a^2z \\ \text{and}\qquad t &= \frac{c}{a} \end{aligned}\right\} \tag{10}$$

Substituting in the first of equations (8) the value of B obtained from the second of equations (9), and the value of $he\,(\theta^2 - \phi^2 z)$ obtained from (7), and making use of (10),

$$p_5 y = -\,8k\,(k^2 - t^2 y) + 44ty^2 + 8tyA\,.$$

Therefore, by the first of equations (9),

$$8kyt^2 + \left(50y^2 - \frac{2}{5}\,p_4 y\right) t = 8k^3 + p_5 y\,. \tag{11}$$

§5. Again, from the last two of equations (9), because $z = e^2 + 1$,

$$he^2\,(\theta^2 + \phi^2 z) = (k^2 + c^2 z) - (2kc + a^3 z).$$

Therefore, by (10),

$$he^2\,(\theta^2 + \phi^2 z) = (k^2 + t^2 y) - a\,(2kt + y)\,. \tag{12}$$

Similarly, from the last two of equations (9),

$$2hze^2\theta\phi = az\,(2kt + y) - (k^2 + t^2 y)\,. \tag{13}$$

And $$(\theta^2 - \phi^2 z)^2 = (\theta^2 + \phi^2 z)^2 - 4z\,(\theta^2\phi^2).$$

Therefore, from (12) and (13),

$$(he^2)^2(\theta^2 - \phi^2 z)^2 = \{\,(k^2 + t^2 y) - a\,(2kt + y)\,\}^2 - \frac{1}{z}\,\{az\,(2kt + y) - (k^2 - t^2 y)\}^2$$

$$\therefore\ zh^2e^2\,(\theta^2 - \phi^2 z)^2 = (k^2 + t^2 y)^2 - y\,(2kt + y)^2.$$

Hence from the value of $he\,(\theta^2 - \phi^2 z)$ in (7),

$$(k^2 + t^2 y)^2 - y\,(2kt + y)^2 = za^2A^2 = yA^2.$$

But, by the first of equations (9),

$$A = \frac{3y}{4} - \frac{p_4}{20}\,. \tag{14}$$

Therefore, $$(k^2 + t^2 y)^2 - y\,(2kt + y)^2 = y\left(\frac{3y}{4} - \frac{p_4}{20}\right)^2. \tag{15}$$

And, from (11),

$$8k\,(k^2 + t^2 y) = (16k^3 + p_5 y) - t\left(50y^2 - \frac{2}{5}\,p_4 y\right).$$

Substitute in (15) the value of $k^2 + t^2 y$ here given. The result is

$$t^2\left\{\left(50y^2 - \frac{2}{5}\,p_4 y\right)^2 - 256k^4 y\right\} + t\left\{-\,2\,(16k^3 + p_5 y)\left(50y^2 - \frac{2}{5}\,p_4 y\right) - 256k^3 y^2\right\}$$

$$= 64k^2 y\left(\frac{3y}{4} - \frac{p_4}{20}\right)^2 + 64k^2 y^3 - (16k^3 + p_5 y)^2. \tag{16}$$

§6. In (11) and (16) we have two equations, with the two unknown quantities y and t. The equations may be written

and

$$\left.\begin{aligned} 8kyt^2 + myt &= n \\ vyt^2 + qyt &= r \end{aligned}\right\} \tag{17}$$

where

$$\begin{aligned} m &= 50y - \frac{2}{5}p_4, \\ n &= 8k^3 + p_5 y, \\ v &= y\left(50y - \frac{2}{5}p_4\right)^2 - 256k^4 \\ &= 2500y^3 - 40p_4y^2 + \frac{4}{25}p_4^2 y - 256k^4, \\ q &= -2(16k^3 + p_5 y)\left(50y - \frac{2}{5}p_4\right) - 256k^3 y \\ &= -100p_5 y^2 + \left(\frac{4}{5}p_4p_5 - 1856k^3\right)y + \frac{64}{5}k^3p_4, \\ r &= 100k^2y^3 - \left(\frac{24p_4k^2}{5} + p_5^2\right)y^2 + \left(\frac{4}{25}p_4^2k^2 - 32k^3p_5\right)y - 256k^6. \end{aligned}$$

§7. The elimination of t from the equations (17) gives us

$$(vn - 8kr)^2 + y\,(qn - mr)(vm - 8kq) = 0. \tag{18}$$

From the values of m, n, v, q and r in §6,

$$\begin{aligned} vn &= 2500p_5y^4 + (20000k^3 - 40p_4p_5)y^3 + \left(\frac{4}{25}p_4^2p_5 - 320p_4k^3\right)y^2 \\ &\quad + \left(\frac{32}{25}p_4^2k^3 - 256k^4p_5\right)y - 2048k^7; \\ 8kr &= 800k^3y^3 - \left(\frac{192}{5}p_4k^3 + 8p_5^2k\right)y^2 + \left(\frac{32}{25}p_4^2k^3 - 256k^4p_5\right)y - 2048k^7; \\ qn &= -100p_5^2y^3 + \left(\frac{4}{5}p_4p_5^2 - 2656k^3p_5\right)y^2 + \left(\frac{96}{5}p_5k^3p_4 - 14848k^6\right)y + 512k^6p_4; \\ mr &= 5000k^2y^4 - (280p_4k^2 + 50p_5^2)y^3 + \left(\frac{248}{25}p_4^2k^2 - 1600k^3p_5 + \frac{2}{5}p_4p_5^2\right)y^2 \\ &\quad - \left(12800k^6 + \frac{8}{125}p_4^3k^2 - \frac{64}{5}k^3p_4p_5\right)y + \frac{512}{5}k^6p_4; \\ vm &= 125000y^4 - 3000p_4y^3 + 24p_4^2y^2 - \left(12800k^4 + \frac{8}{125}p_4^3\right)y + \frac{512}{5}p_4k^4; \\ 8kq &= -800p_5ky^2 + \left(\frac{32}{5}kp_4p_5 - 14848k^4\right)y + \frac{512}{5}p_4k^4. \end{aligned}$$

$$\begin{aligned} \therefore\ vn - 8kr = y^2\Big\{&2500p_5y^2 + (19200k^3 - 40p_4p_5)y \\ &+ \left(\frac{4}{25}p_4^2p_5 - \frac{1048}{5}p_4k^3 + 8p_5^2k\right)\Big\}, \end{aligned}$$

$$\begin{aligned} qn - mr = y \Big\{ & -5000k^2y^3 + (-50p_5^2 + 280p_4k^2)\,y^2 \\ & + \left(\frac{2}{5}p_4p_5^2 - 1056k^3p_5 - \frac{248}{25}p_4^2k^2\right) y \\ & + k^2\left(-2048k^4 + \frac{8}{125}p_4^3 + \frac{32}{5}kp_4p_5\right)\Big\}; \\ vm - 8kq = y \Big\{ & 125000y^3 - 3000p_4y^2 + \left(24p_4^2 + 800kp_5\right) y \\ & - \left(-2048k^4 + \frac{8}{125}p_4^3 + \frac{32}{5}kp_4p_5\right)\Big\}. \end{aligned}$$

By substituting in (18) these values of $vn - 8kr$, $qn - mr$, $vm - 8kq$, we get

$$F(y) = q_0y^6 + q_1y^5 + \dots + q_5y + q_6 = 0, \tag{19}$$

where

$$\begin{aligned} q_0 &= -625000000, \\ q_1 &= 50000000p_4, \\ q_2 &= -200000\,(200kp_5 + 11p_4^2), \\ q_3 &= 102400000k^4 + 1280000kp_4p_5 + 44800p_4^3, \\ q_4 &= -(25600kp_4^2p_5 + 4096000k^4p_4 + 640000p_5^2k^2 + 448p_4^4), \\ q_5 &= 8192k^2p_4p_5^2 - 2048 \times 1856k^5p_5 + 64k^2 + \frac{49152}{5}k^4p_4^2 + \frac{6144}{25}kp_4^3p_5 + \frac{6784}{3125}p_4^5, \\ q_6 &= -\left(\frac{8}{125}p_4^3 - 2048k^4 + \frac{32}{5}kp_4p_5\right)^2. \end{aligned}$$

§8. Assuming now that the coefficients p_3, p_4, p_5 are commensurable quantities, let the commensurable root y of equation (19) be found. Then a^2z is known. Then, from (11) and (16), t or $\frac{c}{a}$ is found.

§9. At this stage, as was indicated in §3, two courses are open to us. One is to proceed to find u_1^5, u_2^5, u_3^5, u_4^5 without troubling ourselves to inquire what a, c, e, θ, ϕ and h are separately. This, the natural and the shortest course, we will now follow. Since a^2z and $\frac{c}{a}$ are known, their product acz is known. And, by (14), A is known. Therefore $czhe\,(\theta^2 - \phi^2z)$, which, by the last of equations (7), is equal to $-aczA$, is known. Hence, by the first of equations (8), B is known. We might even more simply, acz being known, find B from the second of equations (9). The second of equations (8) gives us

$$y\,(B'\sqrt{z}) = 2c\sqrt{z}\,(k^2 - c^2z) + a^2zk\,(a\sqrt{z}) - 2kAa\sqrt{z}. \tag{20}$$

Now acz is known, and it is the same as $(a\sqrt{z})(c\sqrt{z})$; consequently, the signs with which $c\sqrt{z}$ and $a\sqrt{z}$ must be taken relatively to one another are known. Therefore $B'\sqrt{z}$ is known. Therefore $B + B'\sqrt{z}$ is known. But, by the manner in which B, B', B'' and B''' were taken, keeping in view the values of B'' and B''' in (6),

$$u_1^5 = B + B'\sqrt{z} + \sqrt{(hz + h\sqrt{z})},$$

and

$$u_4^5 = B + B'\sqrt{z} - \sqrt{(hz + h\sqrt{z})}.$$

$$\therefore (u_1u_4)^5 = (B + B'\sqrt{z})^2 - (hz + h\sqrt{z}).$$

And, by (7), $u_1u_4 = a\sqrt{z}$. Therefore $hz + h\sqrt{z} = (B + B'\sqrt{z})^2 - (a\sqrt{z})^5$. This gives us

$$\begin{aligned} u_1^5 &= B + B'\sqrt{z} + \sqrt{\{(B + B'\sqrt{z})^2 - (a\sqrt{z})^5\}}, \\ u_4^5 &= B + B'\sqrt{z} - \sqrt{\{(B + B'\sqrt{z})^2 - (a\sqrt{z})^5\}}, \\ u_2^5 &= B - B'\sqrt{z} + \sqrt{\{(B - B'\sqrt{z})^2 + (a\sqrt{z})^5\}}, \\ u_3^5 &= B - B'\sqrt{z} - \sqrt{\{(B - B'\sqrt{z})^2 + (a\sqrt{z})^5\}}. \end{aligned}$$

Hence $u_1 + u_4 + u_2 + u_3$, the root of the given quintic, is known.

To find a, c, e, θ, ϕ and h separately.

§10. If we desire to obtain the values of a, c, e, θ, ϕ and h separately, we may first find a by means of a quadratic equation. By (12) and the third of equations (7),

$$he^2(\theta^2 + \phi^2 z) = (k^2 + t^2 y) - a(2kt + y)$$

and

$$he^2(\theta^2 - \phi^2 z) = -aeA.$$

Therefore,

$$2he^2\theta^2 = (k^2 + t^2 y) - a(2kt + y) - aeA.$$

Also, by (13),

$$2hze^2\theta\phi = az(2kt + y) - (k^2 + t^2 y).$$

Therefore,

$$\frac{\theta}{\phi z} = \frac{(k^2 + t^2 y) - a(2kt + y) - aeA}{az(2kt + y) - (k^2 + t^2 y)}. \tag{21}$$

But, by (9), $B''' = 0$. Therefore, from the last of equations (8),

$$\begin{aligned} \frac{\theta}{\phi z} &= \frac{2e(k^2 - c^2 z) + 2(k^2 + c^2 z) - z(4kc + a^3 z)}{z\{(4kc + a^3 z) - 2(k^2 + c^2 z)\}} \\ &= \frac{2e(k^2 - t^2 y) + 2(k^2 + t^2 y) - az(4kt + y)}{az(4kt + y) - 2z(k^2 + t^2 y)}. \end{aligned} \tag{22}$$

From (21) and (22),

$$\frac{(k^2 + t^2 y) - a(2kt + y) - aeA}{az(2kt + y) - (k^2 + t^2 y)} = \frac{2e(k^2 - t^2 y) + 2(k^2 + t^2 y) - az(4kt + y)}{az(4kt + y) - 2z(k^2 + t^2 y)},$$

$$\therefore \frac{(k^2 + t^2 y) - a(2kt + y) - aeA}{a^2 z(2kt + y) - a(k^2 + t^2 y)} = \frac{2ae(k^2 - t^2 y) + 2a(k^2 + t^2 y) - a^2 z(4kt + y)}{a(a^2 z)(4kt + y) - 2a^2 z(k^2 + t^2 y)},$$

or,

$$\frac{(k^2 + t^2 y) - a(2kt + y) - aeA}{y(2kt + y) - a(k^2 + t^2 y)} = \frac{2ae(k^2 - t^2 y) + 2a(k^2 + t^2 y) - y(4kt + y)}{ay(4kt + y) - 2y(k^2 + t^2 y)}.$$

Put
$$\begin{aligned}\beta &= (k^2 + t^2y) - a(2kt + y),\\ \gamma &= y(2kt + y) - a(k^2 + t^2y),\\ \delta &= ay(4kt + y) - 2y(k^2 + t^2y),\\ \sigma &= 2a(k^2 + t^2y) - y(4kt + y),\\ \tau &= k^2 + t^2y.\end{aligned}$$

Then
$$\frac{\beta - aeA}{\gamma} = \frac{2ae\tau + \sigma}{\delta},$$
$$\therefore ae(\delta A + 2\gamma\tau) = \beta\delta - \gamma\sigma. \tag{23}$$

But $\beta\delta = -a^2y(2kt + y)(4kt + y) + ay(k^2 + t^2y)(8kt + 3y) - 2y(k^2 + t^2y)^2$

and $\gamma\sigma = -\ y^2(2kt + y)(4kt + y) + ay(k^2 + t^2y)(8kt + 3y) - 2a^2(k^2 + t^2y)^2$,

$\therefore \beta\delta - \gamma\sigma = (y - a^2)\{y(2kt + y)(4kt + y) - 2(k^2 + t^2y)^2\}$.

And $y - a^2 = a^2z - a^2 = a^2(z - 1) = a^2e^2$. Therefore

$\beta\delta - \gamma\sigma = a^2e^2\{y(2kt + y)(4kt + y) - 2(k^2 + t^2y)^2\}$.

Therefore, from (23),
$$\delta A + 2\gamma\tau = ae\{y(2kt + y)(4kt + y) - 2(k^2 + t^2y)^2\}, \tag{24}$$
$$\begin{aligned}\therefore (\delta A + 2\gamma\tau)^2 &= a^2e^2\{y(2kt + y)(4kt + y) - 2(k^2 + t^2y)^2\}^2\\ &= (y - a^2)\{y(2kt + y)(4kt + y) - 2(k^2 + t^2y)^2\}^2.\end{aligned} \tag{25}$$

Now k, y and t are known. And, by (14), A is known. Therefore (25) is a quadratic equation from which a can be found. The quadratic has its roots commensurable, and care must be taken in each case to select that one which satisfies all the conditions of the problem. When a has been found, since a^2z and $\frac{c}{a}$ are known, z and c are known; and, because $z = e^2 + 1$, the absolute value of e is known. The sign with which e is to be taken is determined by (24). Next, to find θ, ϕ and h, the third and fourth of equations (9) are $B'' = 1$, $B''' = 0$. Hence, taking the values of B'' and B''' in (8),
$$\left.\begin{aligned}a^2ze &= 2e(k^2 - c^2z)\theta - 2(k^2 + c^2z)(\theta + \phi z) + 2(\theta + \phi)(4kc + a^3z)\\ 0 &= 2e(k^2 - c^2z)\phi + 2(k^2 + c^2z)(\theta + \phi) - (\theta + \phi z)(4kc + a^3z)\end{aligned}\right\} \tag{26}$$

But e, z, a, c, k are known. Therefore, from the simultaneous equations (26), θ and ϕ are known. Therefore, from (7), h is known.

Another way of finding a, c, e, θ, ϕ and h.

§11. The values of a, c, e, θ, ϕ and h may be arrived at in another way. Let B and $B'\sqrt{z}$ be found as in §9. Then, because (see §9)

$$hz + h\sqrt{z} = (B + B'\sqrt{z})^2 - (a\sqrt{z})^5,$$

we have
$$hz = B^2 + (B'\sqrt{z})^2,$$
and
$$h\sqrt{z} = 2B(B'\sqrt{z}) - (a\sqrt{z})^5.$$

Here the quantities to which hz and $h\sqrt{z}$ are equated are known. Therefore h and z are known. When z is known, because a^2z and $\frac{c}{a}$ are known, a and c are known. Finally, θ and ϕ are obtained, as in §10, from the equations (26).

§12. *First Example.*—To exemplify the theory, let us take the equation

$$x^5 + 3x^2 + 2x - 1 = 0. \tag{27}$$

Because $k = -\frac{p_3}{20} = -\frac{3}{20}$, $p_4 = 2$, $p_5 = -1$, the equations (11) and (16) become

$$\left.\begin{aligned} &\frac{6t^2y}{5} + t\left(\frac{4y}{5} - 50y^2\right) = y + \frac{27}{1000}, \\ &t^2\left(\frac{125^2}{9}y^4 - \frac{500y^3}{9} + \frac{4y^2}{9} - \frac{9y}{100}\right) + t\left(\frac{625y^3}{9} + \frac{583}{180}y^2 - \frac{3y}{50}\right) \\ &= \frac{25y^3}{16} - \frac{38y^2}{45} - \frac{13y}{200} - \frac{81}{40000} \end{aligned}\right\} \tag{28}$$

The equation (19), obtained by eliminating t from the equations (28), is

$$F(y) = \left(\frac{25\times125^4}{36}\right)y^6 - \left(\frac{125^4}{9}\right)y^5 + \left(\frac{37\times125^3}{18}\right)y^4 - \left(\frac{1241\times125^2}{90}\right)y^3$$
$$+ \left(\frac{3209\times125}{36}\right)y^2 - \left(\frac{5159}{136}\right)y + \frac{109^2}{22500} = 0,$$

and this has the commensurable root $\frac{1}{125}$. Therefore

$$y = a^2z = \frac{1}{125}.$$

Hence, from the two equations (28),

$$t = \frac{c}{a} = \frac{7}{4}.$$

In subsequent examples, when y and t have been found, we shall proceed at once to find B and $B'\sqrt{z}$, as in §9, without inquiring what a, c, e, θ, ϕ and h are separately. But we desire to illustrate, in one instance, the method of obtaining the values of these quantities which was described in §10, and we will use the present instance for that purpose. To find a we take the equation (25),

$$(\delta A + 2\gamma\tau)^2 = (y - a^2)\{y(2kt + y)(4kt + y) - 2(k^2 + t^2y)^2\}^2. \qquad (29)$$

By (14),

$$A = \frac{3y}{4} - \frac{p_4}{20} = -\frac{47}{500}.$$

Also, from the values of k, y and t above given,

$$2kt + y = -\frac{517}{1000},$$

$$4kt + y = -\frac{521}{1000},$$

$$k^2 + t^2y = \frac{47}{1000}.$$

Therefore

$$\delta = ay(4kt + y) - 2y(k^2 + t^2y) = -\frac{521a + 47}{62500},$$

$$\gamma = y(2kt + y) - a(k^2 + t^2y) = \frac{-47(125a + 11)}{125000},$$

$$\tau = k^2 - t^2y = -\frac{1}{500}.$$

Hence (29) becomes

$$125(323a + 29)^2 = 36^2(1 - 125a^2). \qquad (30)$$

One root of this equation is $-\frac{11}{125}$. But this root proves on examination to be inadmissible. We must therefore take the other root, which is $-\frac{9439}{25^2\times13^2}$. Then, since $c = \frac{7a}{4}$, and $a^2z = \frac{1}{125}$, we have

$$a = -\frac{9439}{25^2\times13^2} = -\frac{9439}{25\times4225},$$

$$c = -\frac{7\times9439}{422500},$$

$$z = \frac{5\times4225^2}{9439^2},$$

$$e = \frac{398}{9439}.$$

The sign of e is determined in the way pointed out in §10. By means of the values of e, z, a, c that have been obtained, we get, from the two equations (26),

$$\theta = \frac{125}{199}, \ \phi = -\frac{18\times9439}{199\times845} \therefore \theta^2 - \phi^2z = -\frac{125}{199}.$$

Therefore, from (7), keeping in view that $A = -\frac{47}{500}$,

$$h = \frac{47}{8}\left(\frac{9439}{13 \times 25 \times 125}\right)^2.$$

Also, from the first two of equations (8), $B = \frac{39}{100}$, and $B' = \frac{91 \times 9439}{500 \times 4225}$

$$\therefore\ B'\sqrt{z} = \frac{91\sqrt{5}}{500},$$

and
$$hz + h\sqrt{z} = \frac{1}{625^2}\left\{\frac{47}{8}\left(21125 + 9439\sqrt{5}\right)\right\}$$

Therefore,

$$\left.\begin{aligned}
u_1^5 &= \frac{13}{100}\left(3 + \frac{7\sqrt{5}}{5}\right) + \frac{1}{625}\sqrt{\left\{\frac{47}{8}\left(21125 + 9439\sqrt{5}\right)\right\}}\\
u_4^5 &= \frac{13}{100}\left(3 + \frac{7\sqrt{5}}{5}\right) - \frac{1}{625}\sqrt{\left\{\frac{47}{8}\left(21125 + 9439\sqrt{5}\right)\right\}}\\
u_2^5 &= \frac{13}{100}\left(3 - \frac{7\sqrt{5}}{5}\right) + \frac{1}{625}\sqrt{\left\{\frac{47}{8}\left(21125 - 9439\sqrt{5}\right)\right\}}\\
u_3^5 &= \frac{13}{100}\left(3 - \frac{7\sqrt{5}}{5}\right) - \frac{1}{625}\sqrt{\left\{\frac{47}{8}\left(21125 - 9439\sqrt{5}\right)\right\}}
\end{aligned}\right\} \qquad (31)$$

Therefore the root of the given quintic is known.

§13. To verify this result, we have $9439\sqrt{5} = 21106.2456$. Therefore

$$\frac{1}{625}\sqrt{\left\{\frac{47}{8}\left(21225 + 9439\sqrt{5}\right)\right\}} = .79696796,$$

$$\frac{1}{625}\sqrt{\left\{\frac{47}{8}\left(21225 - 9439\sqrt{5}\right)\right\}} = .01679462.$$

Also,
$$\frac{13}{100}\left(3 + \frac{7\sqrt{5}}{5}\right) = .79696437$$

and
$$-\frac{13}{100}\left(3 - \frac{7\sqrt{5}}{5}\right) = .01696437.$$

Therefore
$$\begin{aligned}
u_1^5 &= 1.593932\ , & u_1 &= 1.09773\\
-u_4^5 &= .00000359, & -u_4 &= .08147\\
-u_2^5 &= .00016975, & -u_2 &= .17618\\
-u_3^5 &= .03375899, & -u_3 &= .50778.
\end{aligned}$$

Therefore $u_1 + u_2 + u_3 + u_4 = .3323$. Therefore

$$\begin{aligned}
x^5 &= .004052\\
3x^2 &= .331248\\
2x &= \underline{.66448}\\
&\ .99978
\end{aligned}$$

§14. If we wish to exhibit the root as in §9, we find from (8) that

$$B + B'\sqrt{z} = \frac{13}{100}\left(3 + \frac{7\sqrt{5}}{5}\right).$$

Therefore, since $a^5z^2\sqrt{z} = -\frac{\sqrt{5}}{625^2}$,

$$u_1^5 = \frac{13}{100}\left(3 + \frac{7\sqrt{5}}{5}\right) + \sqrt{\left\{\left(\frac{13}{100}\right)^2\left(3 + \frac{7\sqrt{5}}{5}\right)^2 + \frac{\sqrt{5}}{625^2}\right\}},$$

with corresponding expressions for u_2^5, u_3^5, u_4^5. As a matter of fact,

$$\left(\frac{13}{100}\right)^2\left(3 + \frac{7\sqrt{5}}{5}\right) + \frac{\sqrt{5}}{625^2} = \frac{1}{625^2}\left\{\frac{47}{8}\,(21125 + 9439\sqrt{5})\right\}.$$

§15. It is interesting to observe the application of the theory to the equation

$$x^5 - 3x^2 + 2x + 1 = 0, \tag{32}$$

whose roots, with the signs changed, are the same as the roots of the equation (27). By reference to §7, keeping in view that $k = -\frac{p_3}{20}$, it will be seen that, wherever an odd power of p_3 occurs as a factor in a term of any one of the coefficients of the equation $F(y) = 0$, an odd power of p_5 occurs as a factor of the same term. It follows that, by changing the signs of both p_3 and p_5 in the equation (27), in other words, by passing from the equation (27) to the equation (32), $F(y)$ remains unchanged. Therefore the commensurable root of this equation, which we have seen to be $\frac{1}{125}$, is the value of a^2z for the equation (32) as well as for the equation (27). To find t or $\frac{c}{a}$, the equations (17) give us

$$yt = \frac{vn - 8kr}{vm - 8kq}.$$

The values of $vn - 8kr$ and $vm - 8kq$ given in §7 show that, in passing from (27) to (32), $vn - kr$ simply changes its sign, while $vm - kq$ remains unaltered. Hence t or $\frac{c}{a}$ has the same absolute value for the equation (32) as for the equation (27), the signs, however, being different in the two cases. Consequently, for the equation (27), $\frac{c}{a} = -\frac{7}{4}$. Thus we get

$$a = -\frac{9439}{25\times 4225}, \quad c = \frac{7\times 9439}{422500}, \quad z = \frac{5\times 4225^2}{9439^2}, \quad e = -\frac{398}{9439}.$$

Hence, by the first two of equations (8),

$$B + B'\sqrt{z} = -\frac{13}{100}\left(3 + \frac{7\sqrt{5}}{5}\right).$$

Therefore, for the equation (32), the values of u_1^5, u_4^5, u_2^5 and u_3^5 are

$$\begin{aligned}
u_1^5 &= -\frac{13}{100}\left(3 + \frac{7\sqrt{5}}{5}\right) - \sqrt{\left\{\frac{13^2}{100^2}\left(3 + \frac{7\sqrt{5}}{5}\right)^2 + \frac{\sqrt{5}}{625^2}\right\}},\\
u_4^5 &= -\frac{13}{100}\left(3 + \frac{7\sqrt{5}}{5}\right) + \sqrt{\left\{\frac{13^2}{100^2}\left(3 + \frac{7\sqrt{5}}{5}\right)^2 + \frac{\sqrt{5}}{625^2}\right\}},\\
u_2^5 &= -\frac{13}{100}\left(3 - \frac{7\sqrt{5}}{5}\right) - \sqrt{\left\{\frac{13^2}{100^2}\left(3 - \frac{7\sqrt{5}}{5}\right)^2 - \frac{\sqrt{5}}{625^2}\right\}},\\
u_3^5 &= -\frac{13}{100}\left(3 - \frac{7\sqrt{5}}{5}\right) + \sqrt{\left\{\frac{13^2}{100^2}\left(3 - \frac{7\sqrt{5}}{5}\right)^2 - \frac{\sqrt{5}}{625^2}\right\}}.
\end{aligned}$$

p_2 not assumed to be zero.

§17. Let us now consider the more general case in which p_2 is not assumed to be zero. The method that has been illustrated above is still applicable, though the labor of operation, in dealing with particular instances, is increased. Putting, as before, y for a^2z, and t for $\frac{c}{a}$, we form equations corresponding to (11) and (16); from these we obtain the values of y and t; then we find B and $B'\sqrt{z}$ from equations corresponding to the first two of the group (8); or, B can be more readily found from the second of equations (6), and, on the principles of §9, when $B + B'\sqrt{z}$ is known, u_1u_4 or $g^2 - y$ being also known, the root of the given quintic is known.

§18. The values of B and $B'\sqrt{z}$ which correspond, when g or $-\frac{p_2}{10}$ is not zero, to those given in (8) for the case in which g is zero, are obtained from the equations (2) and (3) by keeping in view that, according to the first of equations (4), B is the rational part and B' the coefficient of $\sqrt{z}$ in the expansion of u_1^5.

Put

$$\left.\begin{aligned}
A' &= eh(\theta^2 - \phi^2 z)\\
P &= 2k(k^2 - t^2y) - (g^2 - y)(gk - ty)\\
Q &= 2t(k^2 - t^2y) - (g^2 - y)(k - gt)
\end{aligned}\right\} \qquad (33)$$

Then and

$$\left.\begin{aligned}
(g^2 - y)^2B &= (g^2 + y)P + 2gyQ + 2azA'\{t(g^2 + y) + 2gk\}\\
(g^2 - y)^2B' &= a\{2gP + (g^2 + y)Q\} + 2A'\{k(g^2 + y) + 2gty\}
\end{aligned}\right\} \qquad (34)$$

By (5) and (6)

$$A(g^2-y)=g(k^2-t^2y)+azA',$$

and

$$p_4=-20A+5g^2+15y.$$

Therefore

$$azA'=\frac{g^2-y}{20}(5g^2+15y-p_4)-g(k^2-t^2y). \tag{35}$$

§19. From the second of equations (6),

$$p_5(g^2-y)^2+4B(g^2-y)^2-40ty(g^2-y)^2=0.$$

Therefore, from the first of the two equations (34),

$$(p_5-40ty)(g^2-y)^2+4(g^2+y)P+8gyQ+8azA'\{t(g^2+y)+2gk\}=0.$$

Putting for azA' its value in (35),

$$\begin{gathered}(g^2-y)\left[(p_5-40ty)(g^2-y)+2\left(g^2+3y-\frac{1}{5}p_4\right)\{t(g^2+y)+2gk\}\right]\\ +[4(g^2+y)P+8gazQ-8g(k^2-t^2y)\{t(g^2+y)+2gk\}]=0.\end{gathered} \tag{36}$$

But, from the values of P and Q,

$$\begin{gathered}4(g^2+y)P+8gazQ-8g(k^2-t^2y)\{t(g^2+y)+2gk\}\\ =\{-4(g^2+y)(gk-ty)+8gy(k-gt)-8(k+gt)(k^2-t^2y)\}(g^2-y).\end{gathered}$$

Therefore, rejecting the common factor g^2-y, (36) becomes

$$\begin{gathered}(p_5-40ty)(g^2-y)+2\left(g^2+3y-\frac{1}{5}p_4\right)\{t(g^2+y)+2gk\}\\ -4(g^2+y)(gk-ty)+8gy(k-gt)-8(k+gt)(k^2-t^2y)=0.\end{gathered}$$

Arranging according to the powers of y,

$$\left.\begin{gathered}50y^2t+y\left\{8gt^3+8kt^2-t\left(20g^2+\frac{2}{5}p_4\right)-p_5\right\}\\ +t\left\{2g^2\left(g^2-\frac{p_4}{5}\right)-8gk^2\right\}+g^2p_5-8k^3-\frac{4}{5}gkp_4=0.\end{gathered}\right\} \tag{37}$$

This is one equation between the unknown quantities y and t.

§20. From the last two of the equations (6),

$$he^2(\theta^2+\phi^2z)=(k^2+t^2y)-2kat+(g^2-y)(a-g)$$

and

$$2hze^2(\theta\phi)=2kzat-(k^2+y^2t)-(g^2-y)(az-g).$$

But

$$(\theta^2-\phi^2z)^2=(\theta^2+\phi^2z)^2-4z\theta^2\phi^2.$$

Therefore

$$\begin{aligned}zh^2e^4(\theta^2-\phi^2z)^2=z\{&(k^2+t^2y)-2kat+(g^2-y)(a-g)^2\}\\ &-\{2kzat-(k^2+t^2y)-(g^2-y)(az-g)^2\}.\end{aligned}$$

Therefore

$$z\{he(\theta^2-\phi^2 z)\}^2=(k^2-t^2y)^2+(g^2-y)^3+(g^2-y)\{4kty-2g(k^2+t^2y)\}.$$

But $A'=he(\theta^2-\phi^2 z)$. Therefore, by (36),

$$\begin{aligned}\frac{1}{y}\left\{\frac{g^2-y}{20}(5g^2+15y-p_4)-g(k^2-t^2y)\right\}^2=(k^2-t^2y)^2+(g^2-y)^3&\\ +(g^2-y)\{4kty-2g(k^2+t^2y)\};&\end{aligned}$$

or, arranging according to the powers of y,

$$\begin{aligned}&25y^3-y^2\left(16t^4+56gt^2-64kt+\frac{6}{5}p_4+35g^2\right)\\ &-y\left\{8gt^2\left(g^2-\frac{1}{5}p_4\right)-32k^2t^2+\left(g^2-\frac{1}{5}p_4\right)\left(5g^2+\frac{1}{5}p_4\right)+8gk^2-16g^4\right\}\\ &-g^2\left(g^2-\frac{1}{5}p_4\right)^2+8gk^2\left(g^2-\frac{1}{5}p_4\right)-16k^4=0.\end{aligned}\tag{38}$$

This is the second equation between the unknown quantities y and t.

§21. We may now either eliminate t from the two equations (37) and (38) so as to obtain an equation

$$F(y)=0$$

whose coefficients are rational functions of the coefficients of the quintic to be solved, or we may eliminate y so as to obtain an equation

$$\psi(t)=0$$

whose coefficients are rational functions of those of the quintic to be solved. In the former case, let the commensurable root y of the equation $F(y)=0$ be found. Then, by (37) and (38), t is known. In the latter case, let the commensurable root t of the equation $\psi(t)=0$ be found. Then, by (37) and (38), y is known. When y and t have thus been found, we find B and $B'\surd z$, exactly as in §12, from the equations (34), or B can more readily be found from the second of equations (6). Then

$$\begin{aligned}u_1^5&=B+B'\surd z+\surd\{(B+B'\surd z)^2-(u_1u_4)^5\}\\ &=B+B'\surd z+\surd\{(B+B'\surd z)^2-(g+a\surd z)^5\}.\end{aligned}$$

Therefore

$$\begin{aligned}x&=u_1+u_4+u_2+u_3\\ &=[B+B'\surd z+\surd\{(B+B'\surd z)^2-(g+a\surd z)^5\}]^{\frac{1}{5}}\\ &+[B+B'\surd z-\surd\{(B+B'\surd z)^2-(g+a\surd z)^5\}]^{\frac{1}{5}}\\ &+[B-B'\surd z+\surd\{(B-B'\surd z)^2-(g-a\surd z)^5\}]^{\frac{1}{5}}\\ &+[B-B'\surd z-\surd\{(B-B'\surd z)^2-(g-a\surd z)^5\}]^{\frac{1}{5}}.\end{aligned}$$

It need scarcely be pointed out that since $y=a^2z$, $a\surd z$ is known.

§22. *Second Example.*—As an illustrative example, let

$$x^5-10x^3-20x^2-1505x-7412=0.$$

Here $g=-\frac{p_2}{10}=1$, and $k=-\frac{p_3}{20}=1$. Therefore the equations (37) and (38) become

$$50y^2t+y(8t^3+8t^2+528t+7412)+596t-6216=0,$$

and

$$25y^3-(16t^4+56t^2-64t-1771)y^2-(2384t^2-89384)y-88804=0.$$

The commensurable values of y and t which satisfy these equations are $y=2$, $t=-4$. Then, we can get the values of B and $B'\sqrt{z}$ from the equations (34), keeping in view that azA' is known by (35). As far as B is concerned, it is simpler to make use of the second of equations (6), keeping in view that $acz=ty=-8$. Therefore

$$4B=7412-320,$$
$$\therefore\ B=1773.$$

In order to obtain the value of $B'\sqrt{z}$, we must find P, Q and azA'. By (33) and (35),

$$P=-62+9=-53,$$
$$Q=248+5=253,$$

and

$$azA'=-77+31=-46.$$

Therefore

$$\begin{aligned} B'\sqrt{z} &= 653a\sqrt{z}+2A'\sqrt{z} \\ &= 653a\sqrt{z}-\frac{26\,(azA')\,a\sqrt{z}}{a^2z} \\ &= a\sqrt{z}\,(653+598)=1251\sqrt{2}, \end{aligned}$$
$$\therefore\ B+B'\sqrt{z}=9\,(197+139\sqrt{2}).$$

Hence, since $u_1u_4=g+a\sqrt{z}=1+\sqrt{2}$,

$$\begin{aligned} x &= 9^{\frac{1}{5}}\Big[(197+139\sqrt{2})+\sqrt{\{(197+139\sqrt{2})^2-\tfrac{1}{81}(1+\sqrt{2})^5}\Big]^{\frac{1}{5}} \\ &+ 9^{\frac{1}{5}}\Big[(197+139\sqrt{2})-\sqrt{\{(197+139\sqrt{2})^2-\tfrac{1}{81}(1+\sqrt{2})^5}\Big]^{\frac{1}{5}} \\ &+ 9^{\frac{1}{5}}\Big[(197-139\sqrt{2})+\sqrt{\{(197-139\sqrt{2})^2-\tfrac{1}{81}(1-\sqrt{2})^5}\Big]^{\frac{1}{5}} \\ &+ 9^{\frac{1}{5}}\Big[(197-139\sqrt{2})-\sqrt{\{(197-139\sqrt{2})^2-\tfrac{1}{81}(1-\sqrt{2})^5}\Big]^{\frac{1}{5}}. \end{aligned}$$

To verify this result,

$$\frac{1}{81}(1+\sqrt{2})^5 = 1.012496,$$

$$\frac{1}{81}(1-\sqrt{2})^5 = -.000150535,$$

$$197+139\sqrt{2} \quad = 393.57568,$$

$$(197+139\sqrt{2})^2 = 154901.8,$$

$$197-139\sqrt{2} \quad = .424315,$$

$$(197-139\sqrt{2})^2 = .1800434,$$

$$\sqrt{\left\{(197+139\sqrt{2})^2-\frac{1}{81}(1+\sqrt{2})^5\right\}} = 393.57436,$$

$$\sqrt{\left\{(197-139\sqrt{2})^2-\frac{1}{81}(1-\sqrt{2})^5\right\}} = \quad .180194.$$

Therefore

$$\begin{aligned} u_1 &= 5.888, \\ u_4 &= .412, \\ u_2 &= 1.502, \\ u_3 &= -.276, \\ \therefore\ x &= 7.526. \end{aligned}$$

Modification of the Method to Meet Special Cases.

First special case: When p_2 and p_3 are both zero.

§23. When p_2 and p_3 are both zero, a modification of the general method is rendered necessary by the circumstance that the equations (11) and (16), from which y and t are to be found, are then virtually one, and so are insufficient to give us the values of y and t. In fact, they become

$$\left.\begin{aligned} & t\left(50y^2-\frac{2}{5}p_4y\right)-p_5y=0 \\ \text{and} \qquad & \left\{t\left(50y^2-\frac{2}{5}p_4y\right)-p_5y\right\}^2=0 \end{aligned}\right\} \qquad (39)$$

§24. In an article which appeared in No. 2, Vol. VII of this *Journal*, the present writer showed that, when p_2 and p_3 are both zero, p_4 and p_5 have the forms

$$\left.\begin{aligned} p_4 &= \frac{5n^4(3-m)}{16+m^2} \\ p_5 &= \frac{n^5(22+m)}{16+m^2} \end{aligned}\right\} \qquad (40)$$

These expressions for p_4 and p_5 furnish the criterion of solvability for the quintic

$$x^5 + p_4 x + p_5 = 0. \tag{41}$$

The root of the equation is

$$x = \theta^{\frac{1}{5}} + \alpha\theta^{\frac{2}{5}} + \lambda\alpha^2\theta^{\frac{3}{5}} - \lambda\alpha^3\theta^{\frac{4}{5}},$$

where λ is a root of the quartic equation

$$\lambda^4 - m\lambda^3 - 6\lambda^2 + m\lambda + 1 = 0,$$

and

$$\alpha = -\frac{\lambda^2 + 1}{n\lambda(\lambda - 1)},$$

and

$$\theta = -\frac{n^5\lambda(\lambda - 1)^2}{(16 + m^2)(\lambda + 1)(\lambda^2 + 1)}.$$

In the same issue of the *Journal* in which these results were established, Mr. J. C. Glashan of Ottawa, in "Notes on the Quintic," gave the relations between the coefficients of the solvable quintic

$$x^5 + p_2 x^3 + p_3 x^2 + p_4 x + p_5 = 0,$$

and, in his wider formulæ, the forms of p_4 and p_5 in (40) are included. They were subsequently announced by Mr. Emory McClintock, who had discovered them independently. It is to be regretted that Mr. Glashan has not made public the method by which his conclusions were reached.

§25. From our present position the criterion of solvability of the quintic (41) can be at once deduced, and the solution of the equation effected more readily than by the process employed in the article of the *Journal* just referred to. For, put

$$m = \frac{4A}{y} \text{ and } n = 2t;$$

then the first of the equations (9) may be written

$$p_4 = 5y(3 - m). \tag{42}$$

Also, by the second of equations (9),

$$p_5 = -4B + 40ty. \tag{43}$$

But, from the first of equations (8), $B = -t(y + 2A)$. Therefore

$$p_5 = 44ty + 8At = 2ty(22 + m). \tag{44}$$

And, by (15), in connection with (14),

$$t^4 y - y^2 = A^2 = \frac{m^2y^2}{16},$$

$$\therefore y = \frac{16t^4}{16 + m^2} = \frac{n^4}{16 + m^2}.$$

Therefore also
$$2ty = \frac{2tn^4}{16+m^2} = \frac{n^5}{16+m^2}.$$

By the substitution of these values of y and $2ty$ in (42) and (43), the formulæ (40) are obtained, n in (40) being what we have called $2t$. To find now the root of the equation (41), eliminate m from the equations (40). The result is a sextic equation,
$$\psi(n) = 0.$$

When the coefficients of the quintic (41) are commensurable, the sextic $\psi(n) = 0$ has a commensurable root. Let this be found. Then n is known. Consequently, since $n = 2t$, t is known. Then y is known from (39). Then B is obtained from the second of equations (9), and $B'\sqrt{z}$ from (20). Also
$$u_1u_4 = g + a\sqrt{z} = a\sqrt{z} = \sqrt{y}.$$

Therefore u_1u_4 is known. Therefore, as in §9, the root of the given quintic is found.

§26. *Third Example.*—As an illustrative example, let
$$x^5 + \frac{625}{4}x + 3750 = 0.$$

Here the equations furnishing the criterion of solvability are
$$\frac{625}{4} = \frac{5n^4(3-m)}{16+m^2},$$
$$3750 = \frac{n^5(22+m)}{16+m^2}.$$

These are satisfied by the values $m = 2$, $n = 5$. Therefore
$$t = \frac{5}{2}.$$

Therefore, by (39),
$$y = \frac{125}{4}.$$

Therefore, by the second of equations (9),
$$B = -\frac{625}{4}.$$

And, by (20),
$$yB'\sqrt{z} = -2(c^2z)(c\sqrt{z}) = -2(t^2y)(t\sqrt{y}),$$
$$\therefore\ B'\sqrt{z} = -2t^3\sqrt{y} = -\frac{625}{8}\sqrt{5}.$$
$$\therefore\ B + B'\sqrt{z} = -\frac{625}{4}\left(1 + \frac{\sqrt{5}}{2}\right).$$

And $u_1u_4 = a\sqrt{z} = \frac{5\sqrt{5}}{2}$. Therefore

$$\begin{aligned} x = & \left[-\tfrac{625}{4}\left(1+\tfrac{\sqrt{5}}{2}\right) + \sqrt{\left\{\left(\tfrac{625}{4}\right)^2\left(1+\tfrac{\sqrt{5}}{2}\right)^2 - \left(\tfrac{5\sqrt{5}}{2}\right)^5\right\}}\,\right]^{\frac{1}{5}} \\ & + \left[-\tfrac{625}{4}\left(1+\tfrac{\sqrt{5}}{2}\right) - \sqrt{\left\{\left(\tfrac{625}{4}\right)^2\left(1+\tfrac{\sqrt{5}}{2}\right)^2 - \left(\tfrac{5\sqrt{5}}{2}\right)^5\right\}}\,\right]^{\frac{1}{5}} \\ & + \left[-\tfrac{625}{4}\left(1-\tfrac{\sqrt{5}}{2}\right) + \sqrt{\left\{\left(\tfrac{625}{4}\right)^2\left(1-\tfrac{\sqrt{5}}{2}\right)^2 + \left(\tfrac{5\sqrt{5}}{2}\right)^5\right\}}\,\right]^{\frac{1}{5}} \\ & + \left[-\tfrac{625}{4}\left(1-\tfrac{\sqrt{5}}{2}\right) - \sqrt{\left\{\left(\tfrac{625}{4}\right)^2\left(1-\tfrac{\sqrt{5}}{2}\right)^2 + \left(\tfrac{5\sqrt{5}}{2}\right)^5\right\}}\,\right]^{\frac{1}{5}}. \end{aligned}$$

Second special case: When $u_1u_4 = u_2u_3$.

§27. In the case in which $u_1u_4 = u_2u_3$, $a = 0$. Consequently, if c is distinct from zero, t or $\frac{c}{a}$ is infinite; while, if c is zero, t assumes the form $\frac{0}{0}$. As we cannot here proceed by finding t, the general method has to be modified.

§28. Because $a = 0$, $y = a^2z = 0$, and $ty = \left(\frac{c}{a}\right)(a^2z) = 0$. Also $t^2y = c^2z$. Equation (5) becomes $\quad gA = k^2 - c^2z.$

Therefore, from the first of equations (6),

$$k^2 - c^2z = \frac{g}{20}(5g^2 - p_4), \tag{45}$$

and, from the second of equations (6),

$$p_5 = -4B. \tag{46}$$

From (33) and (35),

$$\begin{aligned} P &= 2k(k^2 - c^2z) - (g^2 - a^2z)(gk - acz) \\ &= 2k(k^2 - c^2z) - g^3k, \end{aligned}$$

and

$$aQ = 2at(k^2 - c^2z) - (g^2 - a^2z)(ak - gc) = g^3c.$$

Therefore, from the first of equations (34), taken in connection with (46),

$$\frac{1}{4}g^2p_5 + k\{2(k^2 - c^2z) - g^3\} + 2czA' = 0. \tag{47}$$

But, A' having been put for $he(\theta^2 - \phi^2z)$, we have, from the value of $z\{he(\theta^2 - \phi^2z)\}$, obtained in §20,

$$z(A')^2 = (k^2 - c^2z)^2 + g^6 - 2g^3(k^2 + c^2z). \tag{48}$$

Therefore, from (47), by the elimination of A',

$$4c^2z = \frac{\left[\frac{1}{4}g^2p_5 + k\{2(k^2 - c^2z) - g^3\}\right]^2}{(k^2 - c^2z)^2 + g^6 - 2g^3(k^2 + c^2z)}. \tag{49}$$

But, by (45), c^2z is known in terms of the coefficients of the given quintic. Therefore the equation (49) gives a relation necessarily subsisting between the coefficients p_2, p_3, p_4 and p_5 of the solvable quintic in which $u_1u_4 = u_2u_3$, in order that u_1u_4 may be equal to u_2u_3. To find now the root of the quintic, c^2z is known by (45), and B is known by (46). To find $B'\sqrt{z}$, making use of the values of P and aQ and $(A'\sqrt{z})^2$ obtained above, we have, from the second of equations (34),

$$g^2(B'\sqrt{z}) = c\sqrt{z}\{2(k^2 - c^2z) + g^3\} + 2k(A'\sqrt{z}). \tag{50}$$

Now, by the first of equations (34), keeping in view the value of azA' in (35),

$$\begin{aligned} g^2B &= g^2\{2k(k^2 - c^2z) - g^3k\} + 2azA'\left(\frac{g^2c}{a}\right) \\ &= g^2\{2k(k^2 - c^2z) - g^3k\} + 2g^2czA'. \end{aligned}$$

As B and c^2z are known, this equation determines the sign of czA' or $(c\sqrt{z})(A'\sqrt{z})$, and therefore determines the signs with which $c\sqrt{z}$ and $A'\sqrt{z}$ are to be taken relatively to one another. Hence $z(A')^2$ being given by (48), $B'\sqrt{z}$ is given by (50). Therefore $B + B'\sqrt{z}$ is known. And $u_1u_4 = g$. Therefore, because

$$u_1^5 = B + B'\sqrt{z} + \sqrt{\{(B + B'\sqrt{z}) - g^5\}},$$

the root of the quintic is known.

§29. *Fourth Example.*—As an illustrative example, let

$$x^5 - \frac{22}{5}x^3 - \frac{11}{25}x^2 + \frac{11 \times 42}{125}x + \frac{11 \times 89}{3125} = 0.$$

Finding the value of c^2z from (45), and substituting in (49), we find that (49) is satisfied. Then, as in the preceding section,

$$p_5 = -4B \ \therefore\ B = -\frac{11 \times 89}{4 \times 5^5}.$$

The value of c^2z is $\frac{5}{16}\left(\frac{11}{25}\right)^2$. Therefore

$$k^2 - c^2z = \left(\frac{11}{25}\right)^2\left(\frac{1}{400} - \frac{5}{16}\right) = -\frac{31}{100}\left(\frac{11}{25}\right)^2,$$

and

$$k^2 + c^2z = \left(\frac{11}{25}\right)^2\left(\frac{1}{400} + \frac{5}{16}\right) = \frac{63}{200}\left(\frac{11}{25}\right)^2.$$

From (48), $$(A'\sqrt{z})^2 = \left(\frac{11^2 \times 5}{4 \times 5^5}\right)^2.$$

The value of czA', obtained from (47), is negative. Therefore $c\sqrt{z}$ and $A'\sqrt{z}$ must have different signs. Put

$$c\sqrt{z} = \frac{11}{25}\,\frac{\sqrt{5}}{4}$$

and $$A'\sqrt{z} = -\frac{1}{5}\left(\frac{11}{25}\right)^2 \frac{\sqrt{5}}{4}.$$

Then, from (50), dividing by g^2,

$$B'\sqrt{z} = -\frac{11\sqrt{5}}{4 \times 5^3}.$$

And $u_1 u_4 = g$. Therefore

$$u_1^5 = \frac{-11}{4 \times 5^5}(89 + 25\sqrt{5}) + \sqrt{\left\{\frac{121}{16 \times 25^5}(89 + 25\sqrt{5})^2 - \left(\frac{11}{25}\right)^5\right\}}.$$

This is, in a different form, the value of u_1^5 given by Lagrange.

§30. *Fifth Example.*—In the instance just considered, c is distinct from zero. It may be well to give an example in which c is zero, as the mode adopted above of determining the sign of $A'\sqrt{z}$ does not apply in that case. Let

$$x^5 + 20x^3 + 20x^2 + 30x + 10 = 0.$$

Here $g = -2$, and $k = -1$. Therefore (45) becomes

$$20(1 - c^2 z) = -2(20 - 30).$$

Therefore $c^2 z = 0$. This value of $c^2 z$ satisfies (49). Then, by (46), $B = -\frac{5}{2}$.

From 48, $$(A'\sqrt{z})^2 = 1 + 64 + 16 = 81 \;\therefore\; A'\sqrt{z} = 9.$$

Because c is zero, we cannot determine the sign of $A'\sqrt{z}$ relatively to that of $c\sqrt{z}$. But the root is the same, whatever sign be taken. Having found $A'\sqrt{z}$, we have, from (50), $4B'\sqrt{z} = -2 \times 9 \;\therefore\; B'\sqrt{z} = -\frac{9}{2}$. Therefore

$$B + B'\sqrt{z} = -\frac{5}{2} - \frac{9}{2} = -7,\quad B - B'\sqrt{z} = -\frac{5}{2} + \frac{9}{2} = 2.$$

And $u_1 u_4 = g = -2$. Therefore

$$\begin{aligned} u_1^5 &= -7 + \sqrt{\{49 - (-2)^5\}} = 2, \\ u_4^5 &= -7 - \sqrt{\{49 - (-2)^5\}} = -16, \\ u_2^5 &= 2 + \sqrt{\{4 + 2^5\}} = 8, \\ u_3^5 &= 2 - \sqrt{\{4 + 2^5\}} = -4. \end{aligned}$$

Therefore $x = 2^{\frac{1}{5}} - 2^{\frac{2}{5}} + 2^{\frac{3}{5}} - 2^{\frac{4}{5}}$.

§31. In the article of the *American Journal of Mathematics* (Vol. VI, page 103) referred to in the opening paragraph of this paper, the solvable irreducible quintic in which u_1u_4 is equal to u_2u_3 was discussed, and the roots of the equation were shown to be determinable in terms of the coefficients p_2, p_3, etc., even while these coefficients have no definite numerical values assigned to them, but remain symbolical. The solution that has now been given is much simpler than the former; equally with the former, it is applicable to equations with symbolical coefficients, the assumption being of course made that the coefficients are related as in (49); and it possesses the advantage of being part of a general theory.

ADDITIONAL EXAMPLES.

§32. *Sixth Example.*—Let

$$x^5 + 320x^2 - 1000x + 4288 = 0.$$

Here $g = 0$, $k = -16$. Because $g = 0$, we use the formulæ (11) and (16). The commensurable values of y and t which satisfy (11) and (16) are

$$y = 8, \quad t = 6.$$

Also, by (14), $$A = 56.$$

Therefore, from the second of equations (9) and from (20),

$$B = -16 \times 37, \quad B'\sqrt{z} = -16 \times 20,$$
$$\therefore B + B'\sqrt{z} = -16(37 + 20\sqrt{2}).$$

And $u_1u_4 = -2\sqrt{2}$. Therefore

$$\begin{aligned} x &= u_1 + u_4 + u_2 + u_3 \\ &= [-16(37 + 20\sqrt{2}) + \sqrt{\{256(37 + 20\sqrt{2})^2 - (-2\sqrt{2})^5\}}]^{\frac{1}{5}} \\ &+ [-16(37 + 20\sqrt{2}) - \sqrt{\{256(37 + 20\sqrt{2})^2 - (-2\sqrt{2})^5\}}]^{\frac{1}{5}} \\ &+ [-16(37 - 20\sqrt{2}) + \sqrt{\{256(37 - 20\sqrt{2})^2 - (2\sqrt{2})^5\}}]^{\frac{1}{5}} \\ &+ [-16(37 - 20\sqrt{2}) - \sqrt{\{256(37 - 20\sqrt{2})^2 - (2\sqrt{2})^5\}}]^{\frac{1}{5}}. \end{aligned}$$

§33. *Seventh Example.*—Let

$$\left(\frac{x}{10}\right)^5 + 40\left(\frac{x}{10}\right)^2 - 69\left(\frac{x}{10}\right) + 108 = 0,$$

or, $$x^5 + 40000x^2 - 690000x + 10800000 = 0.$$

Here $g = 0$, $k = -2000$. Because $g = 0$, we use the formulæ (11) and (16).

The commensurable values of y and t which satisfy (11) and (16) are

$$y = 2000, \quad t = 50.$$

Also, by (14),

$$A = 36000.$$

Therefore, from the second of equations (9) and from (20),

$$B = -1700000, \quad B'\sqrt{z} = -400000\sqrt{5}.$$

And $u_1 u_4 = -20\sqrt{5}$. Therefore

$$\begin{aligned} x &= u_1 + u_4 + u_2 + u_3 \\ &= [-100000(17 + 4\sqrt{5}) + \sqrt{\{100000^2(17 + 4\sqrt{5})^2 + (20\sqrt{5})^5\}}]^{\frac{1}{5}} \\ &+ [-100000(17 + 4\sqrt{5}) - \sqrt{\{100000^2(17 + 4\sqrt{5})^2 + (20\sqrt{5})^5\}}]^{\frac{1}{5}} \\ &+ [-100000(17 - 4\sqrt{5}) + \sqrt{\{100000^2(17 - 4\sqrt{5})^2 - (20\sqrt{5})^5\}}]^{\frac{1}{5}} \\ &+ [-100000(17 - 4\sqrt{5}) + \sqrt{\{100000^2(17 - 4\sqrt{5})^2 - (20\sqrt{5})^5\}}]^{\frac{1}{5}}. \end{aligned}$$

§34. *Eighth Example.*—Let

$$x^5 - 20x^3 + 250x - 400 = 0.$$

Here $g = 2$, $k = 0$. Because g is distinct from zero, we use not the formulæ (11) and (16) as in the two preceding examples, but (37) and (38). The commensurable values of y and t which satisfy (37) and (38) are

$$y = 2, \quad t = -2.$$

Therefore, from the second of equations (6), $B = 60$. The value of azA' is obtained from (35). Thus, since $(A'\sqrt{z})(a\sqrt{z}) = azA'$, and since $a\sqrt{z} = \sqrt{y} = \sqrt{2}$, the value of $A'\sqrt{z}$ is known. Then, by the second of equations (34), $B'\sqrt{z} = 44\sqrt{2}$. Therefore

$$B + B'\sqrt{z} = 4(15 + 11\sqrt{2}):$$

And $u_1 u_4 = g + a\sqrt{z} = 2 + \sqrt{2}$. Therefore

$$\begin{aligned} x &= u_1 + u_4 + u_2 + u_3 \\ &= [4(15 + 11\sqrt{2}) + \sqrt{\{16(15 + 11\sqrt{2})^2 - (2 + \sqrt{2})^5\}}]^{\frac{1}{5}} \\ &+ [4(15 + 11\sqrt{2}) - \sqrt{\{16(15 + 11\sqrt{2})^2 - (2 + \sqrt{2})^5\}}]^{\frac{1}{5}} \\ &+ [4(15 - 11\sqrt{2}) + \sqrt{\{16(15 - 11\sqrt{2})^2 - (2 - \sqrt{2})^5\}}]^{\frac{1}{5}} \\ &+ [4(15 - 11\sqrt{2}) - \sqrt{\{16(15 - 11\sqrt{2})^2 - (2 - \sqrt{2})^5\}}]^{\frac{1}{5}}. \end{aligned}$$

§35. *Ninth Example.*—Let

$$x^5 - 5x^3 + \frac{85}{8}x - \frac{13}{2} = 0.$$

Here $g = \frac{1}{2}$, $k = 0$. Because g is distinct from zero, we use the formulæ (37) and (38). The commensurable values of y and t which satisfy (37) and (38) are

$$y = \frac{1}{8},\ t = -1.$$

Therefore, from the second of equations (6), $B = \frac{3}{8}$. Finding $A'\sqrt{z}$ as in the immediately preceding example, we have, from the second of equations (34), $B'\sqrt{z} = \frac{3}{8}\sqrt{2}$. Therefore

$$B + B'\sqrt{z} = \frac{3}{8}(1 + \sqrt{2}).$$

And $u_1 u_4 = \frac{1}{4}(2 + \sqrt{2})$. Therefore

$$\begin{aligned} x &= u_1 + u_4 + u_2 + u_3 \\ &= \left[\frac{3}{8}(1+\sqrt{2}) + \sqrt{\left\{\frac{9}{64}(1+\sqrt{2})^2 - \frac{1}{4^5}(2+\sqrt{2})^5\right\}}\,\right]^{\frac{1}{5}} \\ &+ \left[\frac{3}{8}(1+\sqrt{2}) - \sqrt{\left\{\frac{9}{64}(1+\sqrt{2})^2 - \frac{1}{4^5}(2+\sqrt{2})^5\right\}}\,\right]^{\frac{1}{5}} \\ &+ \left[\frac{3}{8}(1-\sqrt{2}) + \sqrt{\left\{\frac{9}{64}(1-\sqrt{2})^2 - \frac{1}{4^5}(2-\sqrt{2})^5\right\}}\,\right]^{\frac{1}{5}} \\ &+ \left[\frac{3}{8}(1-\sqrt{2}) - \sqrt{\left\{\frac{9}{64}(1-\sqrt{2})^2 - \frac{1}{4^5}(2-\sqrt{2})^5\right\}}\,\right]^{\frac{1}{5}}. \end{aligned}$$

§36. *Tenth Example.*—Let

$$x^5 + \frac{20x}{17} + \frac{21}{17} = 0.$$

This and the next two examples are intended as additional illustrations of the method to be followed when p_2 and p_3 are both zero. The equations furnishing the criterion of solvability given in §24 are

$$\frac{20}{17} = \frac{5n^4(3-m)}{16+m^2}$$

and

$$\frac{21}{17} = \frac{n^5(22+m)}{16+m^2},$$

and they are satisfied by the commensurable values $m = -1$, $n = 1$. But, by §25, $n = 2t$. Therefore $t = \frac{1}{2}$. Putting $k = 0$ and $t = \frac{1}{2}$, equation (11)

becomes
$$\frac{1}{2}\left(50y-\frac{2}{5}p_4\right)=p_5.$$
$$\therefore\ y=\frac{1}{17}\ \therefore\ a\sqrt{z}=\sqrt{y}=\frac{\sqrt{17}}{17}.$$
Therefore also $c\sqrt{z}=t(a\sqrt{z})=\frac{\sqrt{17}}{34}$. Therefore, by (20),
$$yB'\sqrt{z}=-2(c^2z)(c\sqrt{z})\ \therefore\ B'\sqrt{z}=-\frac{\sqrt{17}}{68}.$$
And, by the second of equations (9), $B=-\frac{1}{68}$. Therefore
$$B+B'\sqrt{z}=-\frac{1}{68}(1+\sqrt{17}).$$
And $u_1u_4=\frac{\sqrt{17}}{17}$. Therefore
$$\begin{aligned} x=&\left[-\frac{1}{68}(1+\sqrt{17})+\sqrt{\left\{\left(\frac{1+\sqrt{17}}{68}\right)^2-\left(\frac{\sqrt{17}}{17}\right)^5\right\}}\right]^{\frac{1}{5}}\\ &+\left[-\frac{1}{68}(1+\sqrt{17})-\sqrt{\left\{\left(\frac{1+\sqrt{17}}{68}\right)^2-\left(\frac{\sqrt{17}}{17}\right)^5\right\}}\right]^{\frac{1}{5}}\\ &+\left[-\frac{1}{68}(1-\sqrt{17})+\sqrt{\left\{\left(\frac{1-\sqrt{17}}{68}\right)^2+\left(\frac{\sqrt{17}}{17}\right)^5\right\}}\right]^{\frac{1}{5}}\\ &+\left[-\frac{1}{68}(1-\sqrt{17})-\sqrt{\left\{\left(\frac{1-\sqrt{17}}{68}\right)^2+\left(\frac{\sqrt{17}}{17}\right)^5\right\}}\right]^{\frac{1}{5}}. \end{aligned}$$

§37. *Eleventh Example.*—Let
$$x^5-\frac{4x}{13}+\frac{29}{65}=0.$$
The equations furnishing the criterion of solvability are
$$-\frac{4}{13}=\frac{5n^4(3-m)}{16+m^2},$$
$$\frac{29}{65}=\frac{n^5(22+m)}{16+m^2},$$
and they are satisfied by the values $m=7$, $n=1$. By §25, $n=2t$. Therefore $t=\frac{1}{2}$. Therefore, from (11), $y=\frac{1}{65}$. Therefore
$$a\sqrt{z}=\frac{\sqrt{65}}{65}\ \therefore\ c\sqrt{z}=t(a\sqrt{z})=\frac{\sqrt{65}}{130}.$$
Therefore, by (20), $yB'\sqrt{z}=-2(c^2z)(c\sqrt{z})$. Therefore $B'\sqrt{z}=-\frac{\sqrt{65}}{260}$. And, by the second of equations (9), $B=-\frac{9}{260}$. Therefore
$$B+B'\sqrt{z}=-\frac{9+\sqrt{65}}{260}.$$

And $u_1u_4 = \dfrac{\sqrt{65}}{65}$. Therefore

$$x = \left[-\frac{9+\sqrt{65}}{260} + \sqrt{\left\{\left(\frac{9+\sqrt{65}}{260}\right)^2 - \left(\frac{\sqrt{65}}{65}\right)^5\right\}}\right]^{\frac{1}{5}}$$

$$+\left[-\frac{9+\sqrt{65}}{260} - \sqrt{\left\{\left(\frac{9+\sqrt{65}}{260}\right)^2 - \left(\frac{\sqrt{65}}{65}\right)^5\right\}}\right]^{\frac{1}{5}}$$

$$+\left[-\frac{9-\sqrt{65}}{260} + \sqrt{\left\{\left(\frac{9-\sqrt{65}}{260}\right)^2 + \left(\frac{\sqrt{65}}{65}\right)^5\right\}}\right]^{\frac{1}{5}}$$

$$+\left[-\frac{9-\sqrt{65}}{260} - \sqrt{\left\{\left(\frac{9-\sqrt{65}}{260}\right)^2 + \left(\frac{\sqrt{65}}{65}\right)^5\right\}}\right]^{\frac{1}{5}}.$$

§38. *Twelfth Example.*—Let

$$x^5 + \frac{10x}{13} + \frac{3}{13} = 0.$$

The equations furnishing the criterion of solvability are

$$\frac{10}{13} = \frac{5n^4(3-m)}{16+m^2},$$

$$\frac{3}{13} = \frac{n^5(22+m)}{16+m^2},$$

and they are satisfied by the values $m = -7$, $n = 1$. By §25, $n = 2t$. Therefore $t = \dfrac{1}{2}$. Therefore, from (11), $y = \dfrac{1}{65}$. Therefore

$$c\sqrt{z} = t\sqrt{y} = \frac{\sqrt{65}}{130}.$$

And, by (20), $yB'\sqrt{z} = -2(c^2z)(c\sqrt{z})$. Therefore $B'\sqrt{z} = -\dfrac{\sqrt{65}}{260}$. And, by the second of equations (9), $B = \dfrac{1}{52}$. Therefore

$$B + B'\sqrt{z} = \frac{5-\sqrt{65}}{260}.$$

And $u_1u_4 = \dfrac{\sqrt{65}}{65}$. Therefore

$$x_1 = \left[\frac{5-\sqrt{65}}{260} + \sqrt{\left\{\left(\frac{5-\sqrt{65}}{260}\right)^2 - \left(\frac{\sqrt{65}}{65}\right)^5\right\}}\right]^{\frac{1}{5}}$$

$$+\left[\frac{5-\sqrt{65}}{260} - \sqrt{\left\{\left(\frac{5-\sqrt{65}}{260}\right)^2 - \left(\frac{\sqrt{65}}{65}\right)^5\right\}}\right]^{\frac{1}{5}}$$

$$+\left[\frac{5+\sqrt{65}}{260} + \sqrt{\left\{\left(\frac{5+\sqrt{65}}{260}\right)^2 + \left(\frac{\sqrt{65}}{65}\right)^5\right\}}\right]^{\frac{1}{5}}$$

$$+\left[\frac{5+\sqrt{65}}{260} - \sqrt{\left\{\left(\frac{5+\sqrt{65}}{260}\right)^2 + \left(\frac{\sqrt{65}}{65}\right)^5\right\}}\right]^{\frac{1}{5}}.$$

§39. *Thirteenth Example.*—Let

$$x^5 + 110\,(5x^3 + 60x^2 + 800x + 8320) = 0.$$

Here $g = -55$, $k = -330$, and the commensurable values of y and t which satisfy (37) and (38) are

$$y = 5 \times 11^2,\ t = 10.$$

Therefore, from the second of equations (6), $B = -220 \times 765$. Finding $A'\sqrt{z}$ as in the second example, we have, from the second of equations (34), $B'\sqrt{z} = -220 \times 337\sqrt{5}$. Therefore

$$B + B'\sqrt{z} = -220\,(765 + 337\sqrt{5}).$$

And $u_1u_4 = -11\,(5 + \sqrt{5})$. Therefore

$$\begin{aligned} x = {} & [-220\,(765 + 337\sqrt{5}) + \sqrt{\{220^2(765 + 337\sqrt{5})^2 - (-55 - 11\sqrt{5})^5\}}]^{\frac{1}{5}} \\ & + [-220\,(765 + 337\sqrt{5}) - \sqrt{\{220^2(765 + 337\sqrt{5})^2 - (-55 - 11\sqrt{5})^5\}}]^{\frac{1}{5}} \\ & + [-220\,(765 - 337\sqrt{5}) + \sqrt{\{220^2(765 - 337\sqrt{5})^2 - (-55 + 11\sqrt{5})^5\}}]^{\frac{1}{5}} \\ & + [-220\,(765 - 337\sqrt{5}) - \sqrt{\{220^2(765 - 337\sqrt{5})^2 - (-55 + 11\sqrt{5})^5\}}]^{\frac{1}{5}}. \end{aligned}$$

§40. *Fourteenth Example.*—Let

$$x^5 - 20x^3 - 80x^2 - 150x - 656 = 0.$$

Here $g = 2$, $k = 4$, and the commensurable values of y and t which satisfy (37) and (38) are $y = 2,\ t = 2.$

Therefore, from the second of equations (6), $B = 204$. Finding $A'\sqrt{z}$ as in the immediately preceding example, we have, from the second of equations (34), $B'\sqrt{z} = 144\sqrt{2}$. Therefore

$$B = 12 \times 17, \text{ and } B'\sqrt{z} = 144\sqrt{2},$$
$$\therefore\ B + B'\sqrt{z} = 12\,(17 + 12\sqrt{2}).$$

And $u_1u_4 = 2 + \sqrt{2}$. Therefore

$$\begin{aligned} x_1 &= u_1 + u_4 + u_2 + u_3 \\ &= [12\,(17 + 12\sqrt{2}) + \sqrt{\{144\,(17 + 12\sqrt{2})^2 - (2 + \sqrt{2})^5\}}]^{\frac{1}{5}} \\ &\quad + [12\,(17 + 12\sqrt{2}) - \sqrt{\{144\,(17 + 12\sqrt{2})^2 - (2 + \sqrt{2})^5\}}]^{\frac{1}{5}} \\ &\quad + [12\,(17 - 12\sqrt{2}) + \sqrt{\{144\,(17 - 12\sqrt{2})^2 - (2 - \sqrt{2})^5\}}]^{\frac{1}{5}} \\ &\quad + [12\,(17 - 12\sqrt{2}) - \sqrt{\{144\,(17 - 12\sqrt{2})^2 - (2 - \sqrt{2})^5\}}]^{\frac{1}{5}}. \end{aligned}$$

§41. *Fifteenth Example.*—Let

$$x^5 - 40x^3 + 160x^2 + 1000x - 5888 = 0.$$

Here $g = 4$, $k = -8$, and the commensurable values of y and t which satisfy (37) and (38) are $$y = 8, \quad t = -4.$$

Therefore, from the second of equations (6), $B = 1152$. Finding $A'\sqrt{z}$ as in the two preceding examples, we have, from the second of equations (34), $B'\sqrt{z} = 816\sqrt{2}$. Therefore
$$B + B'\sqrt{z} = 48\,(24 + 17\sqrt{2}).$$

And $u_1u_4 = 4 + 2\sqrt{2}$. Therefore
$$\begin{aligned} x = {} & [48\,(24 + 17\sqrt{2}) + \sqrt{\{48^2\,(24 + 17\sqrt{2})^2 - (4 + 2\sqrt{2})^5\}}]^{\frac{1}{5}} \\ & + [48\,(24 + 17\sqrt{2}) - \sqrt{\{48^2\,(24 + 17\sqrt{2})^2 - (4 + 2\sqrt{2})^5\}}]^{\frac{1}{5}} \\ & + [48\,(24 - 17\sqrt{2}) + \sqrt{\{48^2\,(24 - 17\sqrt{2})^2 - (4 - 2\sqrt{2})^5\}}]^{\frac{1}{5}} \\ & + [48\,(24 - 17\sqrt{2}) - \sqrt{\{48^2\,(24 - 17\sqrt{2})^2 - (4 - 2\sqrt{2})^5\}}]^{\frac{1}{5}}. \end{aligned}$$

§42. *Sixteenth Example.*—Let
$$\left(\frac{x}{2}\right)^5 - 50\left(\frac{x}{2}\right)^3 - 600\left(\frac{x}{2}\right)^2 - 2000\left(\frac{x}{2}\right) - 11200 = 0,$$
or
$$x^5 - 200x^3 - 4800x^2 - 32000x - 3200 \times 112 = 0.$$

Here $g = 20$, $k = 240$, and the commensurable values of y and t which satisfy (37) and (38) are $$y = 80, \quad t = 20.$$

Therefore, from the second of equations (6), $B = 640 \times 165$. Finding $A'\sqrt{z}$ as in the preceding examples of the same type, we have, from the second of equations (34), $B'\sqrt{z} = 640 \times 73\sqrt{5}$. Therefore
$$B + B'\sqrt{z} = 640\,(165 + 73\sqrt{5}).$$

And $u_1u_4 = 4\,(5 + \sqrt{5})$. Therefore
$$\begin{aligned} x = {} & [640\,(165 + 73\sqrt{5}) + \sqrt{\{640^2\,(165 + 73\sqrt{5})^2 - (20 + 4\sqrt{5})^5\}}]^{\frac{1}{5}} \\ & + [640\,(165 + 73\sqrt{5}) - \sqrt{\{640^2\,(165 + 73\sqrt{5})^2 - (20 + 4\sqrt{5})^5\}}]^{\frac{1}{5}} \\ & + [640\,(165 - 73\sqrt{5}) + \sqrt{\{640^2\,(165 - 73\sqrt{5})^2 - (20 - 4\sqrt{5})^5\}}]^{\frac{1}{5}} \\ & + [640\,(165 - 73\sqrt{5}) - \sqrt{\{640^2\,(165 - 73\sqrt{5})^2 - (20 - 4\sqrt{5})^5\}}]^{\frac{1}{5}}. \end{aligned}$$

§43. *Seventeenth Example.*—Let
$$x^5 + 110\,(5x^3 + 20x^2 - 360x + 800) = 0.$$

Here $g = -55$, $k = -110$, and the commensurable values of y and t which satisfy (37) and (38) are
$$y = 5 \times 11^2, \quad t = -10.$$

Therefore, from the second of equations (6), $B = -11 \times 7500$. Finding $A'\sqrt{z}$ as in preceding examples of the same type, we have, from the second of equations (34), $B'\sqrt{z} = -11 \times 2700\sqrt{5}$. Therefore

$$B + B'\sqrt{z} = -3300(25 + 9\sqrt{5}).$$

And $u_1u_4 = -11(5 + \sqrt{5})$. Therefore

$$\begin{aligned} x = &[-3300(25 + 9\sqrt{5}) + \sqrt{\{3300^2(25 + 9\sqrt{5})^2 + 11^5(5 + \sqrt{5})^5\}}]^{\frac{1}{5}} \\ &+ [-3300(25 + 9\sqrt{5}) - \sqrt{\{3300^2(25 + 9\sqrt{5})^2 + 11^5(5 + \sqrt{5})^5\}}]^{\frac{1}{5}} \\ &+ [-3300(25 - 9\sqrt{5}) + \sqrt{\{3300^2(25 - 9\sqrt{5})^2 + 11^5(5 - \sqrt{5})^5\}}]^{\frac{1}{5}} \\ &+ [-3300(25 - 9\sqrt{5}) - \sqrt{\{3300^2(25 - 9\sqrt{5})^2 + 11^5(5 - \sqrt{5})^5\}}]^{\frac{1}{5}}. \end{aligned}$$

§44. *Eighteenth Example.*—Let

$$x^5 - 20x^3 + 320x^2 + 540x + 638 = 0.$$

Here $g = 2$, $k = -16$, and the commensurable values of y and t that satisfy (37) and (38) are $y = 8$, $t = -5$.

Therefore, from the second of equations (6), and the second of equations (34),

$$B = -12 \times 166, \quad B'\sqrt{z} = -12 \times 117\sqrt{2},$$
$$\therefore B + B'\sqrt{z} = -12(166 + 117\sqrt{2}).$$

And $u_1u_4 = 2(1 + \sqrt{2})$. Therefore

$$\begin{aligned} x = &[-12(166 + 117\sqrt{2}) + \sqrt{\{144(166 + 117\sqrt{2})^2 - 32(1 + \sqrt{2})^5\}}]^{\frac{1}{5}} \\ &+ [-12(166 + 117\sqrt{2}) - \sqrt{\{144(166 + 117\sqrt{2})^2 - 32(1 + \sqrt{2})^5\}}]^{\frac{1}{5}} \\ &+ [-12(166 - 117\sqrt{2}) + \sqrt{\{144(166 - 117\sqrt{2})^2 - 32(1 - \sqrt{2})^5\}}]^{\frac{1}{5}} \\ &+ [-12(166 - 117\sqrt{2}) - \sqrt{\{144(166 - 117\sqrt{2})^2 - 32(1 - \sqrt{2})^5\}}]^{\frac{1}{5}}. \end{aligned}$$

§45. *Nineteenth Example.*—Let

$$x^5 - 20x^3 - 160x^2 - 420x - 8928 = 0.$$

Here $g = 2$, $k = 8$, and the commensurable values of y and t which satisfy (37) and (38) are $y = 72$, $t = -\frac{7}{3}$.

Therefore, from the second equations (6), $B = 552$. Finding $A'\sqrt{z}$ as in preceding examples of the same type, we have, from the second of equations (34), $B'\sqrt{z} = -284\sqrt{2}$. Therefore

$$B + B'\sqrt{z} = 4(138 - 71\sqrt{2}).$$

And $u_1u_4 = 2 - 6\sqrt{2}$. Therefore

$$\begin{aligned} x = {} & [4(138 - 71\sqrt{2}) + \sqrt{\{16(138 - 71\sqrt{2})^2 - (2 - 6\sqrt{2})^5\}}]^{\frac{1}{5}} \\ & + [4(138 - 71\sqrt{2}) - \sqrt{\{16(138 - 71\sqrt{2})^2 - (2 - 6\sqrt{2})^5\}}]^{\frac{1}{5}} \\ & + [4(138 + 71\sqrt{2}) + \sqrt{\{16(138 + 71\sqrt{2})^2 - (2 + 6\sqrt{2})^5\}}]^{\frac{1}{5}} \\ & + [4(138 + 71\sqrt{2}) - \sqrt{\{16(138 + 71\sqrt{2})^2 - (2 + 6\sqrt{2})^5\}}]^{\frac{1}{5}}. \end{aligned}$$

§46. *Twentieth Example.*—Let

$$x^5 - 20x^3 + 170x + 208 = 0.$$

Here $g = 2$, $k = 0$, and the commensurable values of y and t which satisfy (37) and (38) are

$$y = 2,\ t = 2.$$

Then in the usual way we get

$$B + B'\sqrt{z} = -12(1 - \sqrt{2}).$$

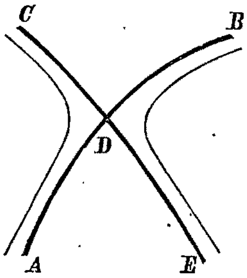
And $u_1u_4 = 2 + \sqrt{2}$. Therefore

$$\begin{aligned} x = {} & [-12(1 - \sqrt{2}) + \sqrt{\{144(1 - \sqrt{2})^2 - (2 + \sqrt{2})^5\}}]^{\frac{1}{5}} \\ & + [-12(1 - \sqrt{2}) - \sqrt{\{144(1 - \sqrt{2})^2 - (2 + \sqrt{2})^5\}}]^{\frac{1}{5}} \\ & + [-12(1 + \sqrt{2}) + \sqrt{\{144(1 + \sqrt{2})^2 - (2 - \sqrt{2})^5\}}]^{\frac{1}{5}} \\ & + [-12(1 + \sqrt{2}) - \sqrt{\{144(1 + \sqrt{2})^2 - (2 - \sqrt{2})^5\}}]^{\frac{1}{5}}. \end{aligned}$$

Forms of Non-Singular Quintic Curves.

By David Barcroft.

Before noticing the forms of the curves given in this paper, I shall make a general statement of the principles on which their construction is based, together with some facts concerning inflections and bitangents of curves in general when derived by the method here considered. The construction of the curves is founded on the possibility of deriving one curve from another by a change of parameters. Here the only parametric variation employed is one involving the absolute term alone of the equation representing the curve, which, in every case, is assumed to be an improper curve composed of two non-singular curves. The equation of such a curve is $uv = 0$, and that of the derived curve $uv = k$. If k is properly chosen, the derived curve will be non-singular, and will be as near as we please to the original. This fact enables us to determine its form if we know the character of the changes brought about by a parametric variation of the kind in question. What these changes are can be readily determined.



Let D be a double point of $uv = 0$. For any particular value of k the product uv must preserve the same sign, namely, the sign of k; u and v, therefore, must change together if they change at all. A simultaneous change can take place only when we pass from an angle ADC to its opposite angle BDE.

If the sign of k be changed, parts of the curve will be in the opposite angles ADE and CDB.

To investigate the nature of $uv = k$ as to the direction of its points relative to $uv = 0$, let x and y be the coordinates of any point on $uv = 0$, and ε_1 and ε_2 the increments of x and y necessary to give a point on $uv = k$; substituting $x + \varepsilon_1$ for x, and $y + \varepsilon_2$ for y, in $uv = 0$, we have

$$\left(u + \varepsilon_1 \frac{\partial u}{\partial x} + \varepsilon_2 \frac{\partial u}{\partial y} + \text{etc.}\right)\left(v + \varepsilon_1 \frac{\partial v}{\partial x} + \varepsilon_2 \frac{\partial v}{\partial y} + \text{etc.}\right) = k.$$

If the point (x, y) is on only one of the curves $u = 0$, $v = 0$, say $u = 0$, leaving out higher powers of ε_1 and ε_2, the result becomes

$$\left(\varepsilon_1 \frac{\partial u}{\partial x} + \varepsilon_2 \frac{\partial u}{\partial y}\right) v = k,$$

the equation of a line parallel to the tangent to the curve $u = 0$ at the point (x, y). This line gives the direction of an element of the curve $uv = k$. If (x, y) is a double point on $uv = 0$, our result will be, under the conditions prescribed above,
$$\left(\varepsilon_1 \frac{\partial u}{\partial x} + \varepsilon_2 \frac{\partial u}{\partial y}\right)\left(\varepsilon_1 \frac{\partial v}{\partial x} + \varepsilon_2 \frac{\partial v}{\partial y}\right) = k,$$

which shows that in the neighborhood of a double point of $uv = 0$, $uv = k$ approximates to an hyperbola whose asymptotes are the tangents at the double point in question.

In the cases of the cubic and quartic, the maximum number of real inflections is one-third of the total number. The conic having no inflections, may be included as possessing the same property. All the quintics whose forms I give have fifteen real inflections, and it is true generally that the maximum number of real inflections of curves formed as these are is one-third of the total number. Let it be granted that the number of real inflections of a curve $u = 0$ of degree n_1 is $n_1 (n_1 - 2)$, and of a curve $v = 0$ of degree n_2 is $n_2 (n_2 - 2)$; let $u = 0$ and $v = 0$ intersect in the maximum number of points $n_1 n_2$, then the curve $uv = k$, k being small enough, will have $(n_1 + n_2)(n_1 + n_2 - 2)$ real inflections; for the inflections of $uv = k$ are those due to the inflections of $uv = 0$ together with two for each double point of $uv = 0$; that is, $n_1 (n_1 - 2) + n_2 (n_2 - 2) + 2 n_1 n_2$, which equals the number just given. In view of the statement made regarding the conic and cubic, the property noticed is seen to be true generally. We may state, therefore, that a curve of any degree n may have as many as $n (n - 2)$

real inflections. It will be observed that, as the intersections of $u = 0$ and $v = 0$ diminish in sets of two, the number of real inflections will diminish in sets of four; hence, if we form a complete system of curves by this method, the number of inflections of the curves of various classes will be $n(n-2)$, $n(n-2)-4$, $n(n-2)-8$, etc.

The bitangents of $uv = k$ may be considered as derived from those of $uv = 0$, including among the latter the analytical bitangents; these are (a), lines drawn from a double point of $uv = 0$ tangent to the curve; (b), lines joining the double points two and two. Each of the analytical bitangents of class (a) gives rise to two, and each of class (b) to four bitangents of $uv = k$; each ordinary bitangent of $uv = 0$ is the source of one of $uv = k$. A figure will make this clear. It will be observed that bitangents derived from analytical bitangents may be imaginary. The total number is made up of

$$\frac{1}{2}(n^4 - 2n^3 - 9n^2 + 18n) - 2n_1n_2\left[n_1(n_1 - 1) - n_1n_2 - 5\right],$$

derived from the ordinary bitangents of $uv = 0$; $2n_1n_2\left[n(n-1) - 2n_1n_2 - 4\right]$, derived from the analytical bitangents of class (a), and $2n_1n_2(n_1n_2 - 1)$ from those of class (b), making together $\frac{1}{2}(n^4 - 2n^3 - 9n^2 + 18n)$, which is the number for a non-singular curve of degree n. n_1 and n_2 represent the orders of $u = 0$ and $v = 0$.

Zeuthen has remarked (Math. Annalen, Vol. 7, p. 410) that the branches that make up a plane curve may be of two kinds, odd or even; an odd branch intersecting any other odd branch in an odd number of points, and an even branch intersecting any other branch, odd or even, in an even number of points. Möbius has shown that an odd branch has always an odd number of real inflections and an even branch an even number, and if an odd branch is non-singular, it must have at least three real inflections. Applying these statements to the quintic, it is seen that a non-singular quintic cannot have more than one odd branch. It must have one with at least three real inflections. It cannot have more than six even branches external to each other, for, if it had as many as seven, we could pass a cubic through seven points chosen on these and two on the odd branch intersecting the quintic in at least seventeen points. There cannot be more than two even branches if one is inside of the other, and, moreover, the internal branch must be an oval.

Zeuthen divides bitangents into two kinds. His bitangents of the first kind are those whose points of contact are on the same branch; the points of contact may be imaginary. Bitangents of the second kind have their points of contact on different branches. In the case of quartics, the points of contact of a bitangent of the first kind cannot be separated by its intersections with two other bitangents. This is true of quintics if the bitangent has its points of contact on an even branch, but the contrary may be the case if the points of contact are on the odd branch. We may have an inflectional tangent, tangent elsewhere. The number of bitangents of the first kind to an even branch is easily seen to be one-half of the number of its real inflections. Quintics may have 0, 1, 2, 3, 4, 5 or 6 even branches, hence the number of bitangents of the second kind to even branches may be 0, 0, 4, 12, 24, 40, or 60, since every combination of even branches in sets of two has four bitangents of the second kind.

The quintics whose forms are given are derived in every case from the combination of a cubic and conic intersecting in six points. Such combinations give curves with fifteen real inflections, and they illustrate, to a certain extent, the forms of quintics having a less number of inflections, especially those resulting from cubics and conics intersecting in four, two, and zero points. In the following I give forms so far as I have been able to determine them. After Zeuthen an even branch is called an oval, a unifolium, a bifolium, a trifolium, etc., according as it has 0, 2, 4, 6, etc. $2r$ real inflections.

In the following enumeration of forms, I put in the same class quintics having the same number of real inflections, then subdivide according to the number of inflections on the even branches. The odd branches are not mentioned, but must be understood as forming a part of the quintic in every case. Each figure represents two forms, in some cases duplicates. To distinguish between them they have been drawn in heavy and light lines. The dotted curves indicate the combination of cubic and conic from which particular forms are derived. The positions of the inflections on the cubics are shown by the dashes drawn across them.

Quintics with Fifteen Real Inflections.

I.

1. Sexifolium and internal oval. Figs. 1, 4.
2. Six unifolia. Figs. 1, 4.

II.

3. A quadrifolium and a unifolium. Fig. 32.
4. A quadrifolium, a unifolium, and an oval. Fig. 33.
5. A trifolium and a bifolium. Fig. 5.
6. A trifolium, a bifolium, and an oval. Fig. 34.
7. A quintifolium and two ovals. Fig. 8.
8. A bifolium and three unifolia. Figs. 2, 9, 13, 14, 17.
9. A bifolium, three unifolia, and an oval. Figs. 22, 31.
10. Two bifolia, a unifolium, and two ovals. Fig. 23.
11. Five unifolia. Figs. 24, 26, 27.

III.

12. A quadrifolium and an oval. Figs. 35, 44.
13. A quadrifolium and two ovals. Fig. 15.
14. A trifolium and a unifolium. Figs. 11, 12.
15. A trifolium, a unifolium, and an oval. Figs. 18, 36.
16. Two bifolia. Fig. 16.
17. Two bifolia and an oval. Fig. 37.
18. Two bifolia and two ovals. Figs. 19, 28.
19. A bifolium and two unifolia. Figs. 38, 39, 41, 42, 45.
20. Four unifolia. Figs. 3, 10, 20.
21. A quadrifolium and three ovals.
22. A bifolium, two unifolia, and an oval.

The figures for 21 and 22 are not given; 21 results from a combination of a bipartite redundant hyperbola with a hyperbola; 22 from the same cubic with an ellipse; Figs. 41, 42, 45 illustrate this case.

IV.

23. A trifolium and internal oval. Fig. not drawn. The form differs from the next only in the internal oval.
24. A trifolium. Figs. 40, 43, 46.
25. A trifolium and an oval. Fig. 40.
26. A trifolium and two ovals. Figs. 7, 39.

27. A trifolium and three ovals. Fig. 12.
28. A bifolium and a unifolium. Fig. 6.
29. A bifolium, a unifolium, and an oval. Figs. 25, 29, 30.
30. Three unifolia. Figs. 25, 29, 30.
31. Three unifolia and an oval. Fig. 11.

V.

32. A bifolium. Figs. 7, 8, 15, 22, 23, 24, 26, 27, 31, 44.
33. A bifolium and an oval. Figs. 2, 14, 21.
34. Two unifolia. Fig. 47.
35. A bifolium and three ovals.
36. Two unifolia and two ovals.

The last two may be obtained by combining a bipartite redundant hyperbola with an ellipse.

VI.

37. A unifolium and an oval. Figs. 3, 10, 20.
38. A unifolium and three ovals.

The last is formed as 35 and 36 by changing the position of the ellipse. The bipartite cubic corresponding to that of Fig. 43 is the proper one for the cases 35, 36, 38.

VII.

39. An oval. Figs. 18, 33, 34, 36.
40. Two ovals. Figs. 9, 13, 16, 17, 41, 42, 44, 45.
41. Three ovals. Figs. 40, 43, 46.
42. Four ovals.

By changing the position of the ellipse in case 38, 42 is obtained.

43. No even branches. Figs. 28, 47.

Quintics with Eleven Real Inflections.

I.

1. A quadrifolium and an internal oval.
2. A quadrifolium and an oval.
3. Two bifolia and an oval.
4. Four unifolia.
5. A trifolium and a unifolium.
6. A trifolium, a unifolium, and an oval.

II.

7. A trifolium.
8. A trifolium and an oval.
9. A trifolium and two ovals.
10. A trifolium and three ovals.
11. A bifolium and a unifolium.
12. A bifolium, a unifolium, and an oval.
13. A bifolium, a unifolium, and two ovals.
14. Three unifolia.
15. Three unifolia and an oval.

III.

16. A bifolium and an internal oval.
17. A bifolium and an oval.
18. A bifolium.
19. Two unifolia.

IV.

20. A unifolium and an oval.

V.

21. An oval.
22. Two ovals.
23. Three ovals.
24. No even branches.

Quintics with Seven Real Inflections.

I.

1. A bifolium with internal oval.
2. A bifolium and an oval.
3. A bifolium.
4. Two unifolia.

II.

5. A unifolium with internal oval.
6. A unifolium and an oval.
7. A unifolium.

III.

8. An oval.
9. Two ovals.
10. No even branches.

Quintics with Three Real Inflections.

I.

1. An oval with internal oval.
2. Two ovals.
3. An oval.

The total number of forms enumerated is eighty.

An inspection of the figures will show that the odd branches are composed of one, three, or five infinite parts with a number of inflections varying from three to fifteen. It will be observed that in the neighborhood of a double point of the improper curve there are always two inflections of the non-singular curve; they are both in the same angle if the parts of the improper curve turned towards each other are convex; one in each of two opposite angles if the parts turned towards each other are, one concave and the other convex. Other inflections are those in the neighborhood of the inflections of the cubic.

With regard to the real bitangents I am unable to state a general law as to their number. It is certain, however, that the number of bitangents of the first kind is not constant. An examination of the curves will show this to be the case. Figs. 13 and 14, 26 and 27 illustrate the loss of two bitangents of the first kind. Intermediate between the heavy curve of Fig. 13 and the light curve of Fig. 14 there is one having an inflectional bitangent. The loss of two bitangents occurs in passing through this case. It will be noticed that two bitangents of the second kind are lost in the case of the other curves of these figures. The light curves of Figs. 6, 7, and 8, and the heavy curve of Fig. 5, illustrate the statement made that the points of contact of a bitangent of the first kind to an odd branch may be separated by the intersections of the bitangent with two others.

I add the following table showing the character of the even branches as they occur in the figures. 0 represents an oval; 1, a unifolium; 2, a bifolium, etc.; (0) indicates that the oval is inside of another even branch. By a proper combination of digits the number and characters of the even branches are shown in each case. *h* refers to the heavy curve and *l* to the light one.

Figs.	1	2	3	4	5	6
h	6, (0);	2, 0;	1, 0;	6, (0);	0, 0;	2, 1;
l	1, 1, 1, 1, 1, 1;	2, 1, 1, 1;	1, 1, 1, 1;	1, 1, 1, 1, 1, 1;	3, 2;	0, 0;

Figs.	7	8	9	10	11	12
h	2;	5, 0, 0;	2, 1, 1, 1;	1, 0;	3, 1;	3, 1;
l	3, 0, 0;	2;	0, 0;	1, 1, 1, 1;	1, 1, 1, 0;	3, 0, 0, 0;

Figs.	13	14	15	16	17	18
h	2, 1, 1, 1;	2, 0;	2;	0, 0;	2, 1, 1, 1;	0;
l	0, 0;	2, 1, 1, 1;	4, 0, 0;	2, 2;	0, 0;	3, 1, 0;

Figs.	19	20	21	22	23	24
h	2, 2, 0, 0;	1, 0;	2, 0;	2;	2;	2;
l	2, 2, 0, 0;	1, 1, 1, 1;	3, 0;	2, 1, 1, 1, 0;	2, 2, 1, 0, 0;	1, 1, 1, 1, 1;

Figs.	25	26	27	28	29	30
h	2, 1, 1;	2;	2;		2, 1, 0;	2, 1, 0;
l	1, 1, 1;	1, 1, 1, 1, 1;	1, 1, 1, 1, 1;	2, 2, 0, 0;	1, 1, 1;	1, 1, 1;

Figs.	31	32	33	34	35	36
h	2;	4, 1;	0;	0;	4, 0;	3, 1, 0;
l	2, 1, 1, 1, 0;	4, 1;	4, 1, 0;	3, 2, 0;	4, 0;	0;

Figs.	37	38	39	40	41	42
h	2, 2, 0;	2, 1, 1;	3, 0, 0;	3;	0, 0;	0, 0;
l	2, 2, 0;	2, 1, 1;	2, 1, 1;	0, 0, 0;	2, 1, 1;	2, 1, 1;

Figs.	43	44	45	46	47
h	0, 0, 0;	4, 0;	0, 0;	3;	
l	3;	2;	2, 1, 1;	0, 0, 0;	1, 1.

I wish to state, in conclusion, that I am indebted to Dr. Story, of the Johns Hopkins University, for most of the matter of the preceding pages. A number of the curves also were pointed out by him, and, in carrying out the work, I have derived much assistance from his suggestions.

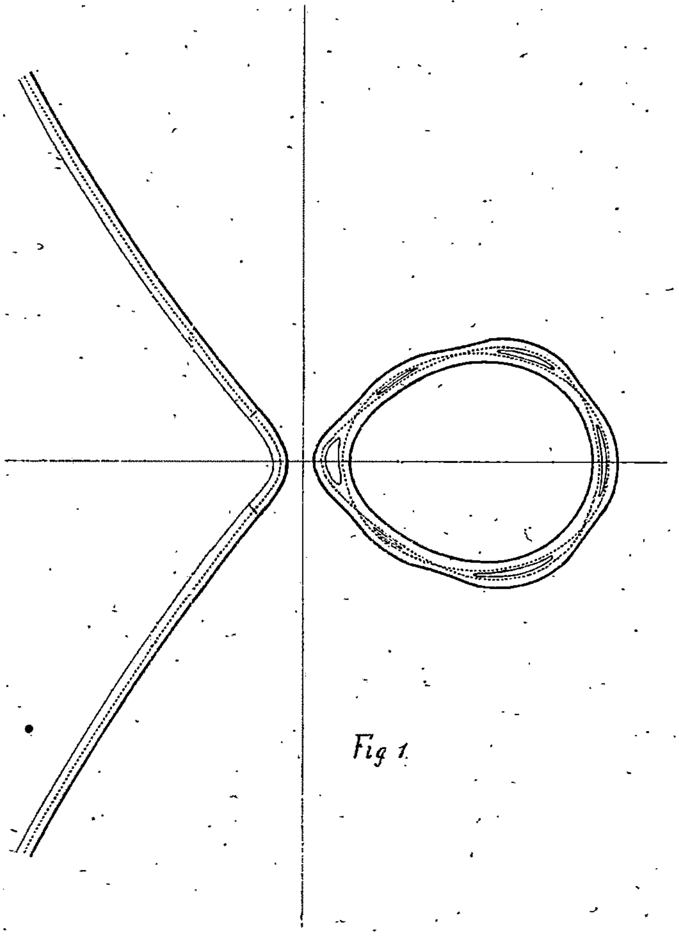


Fig 1.

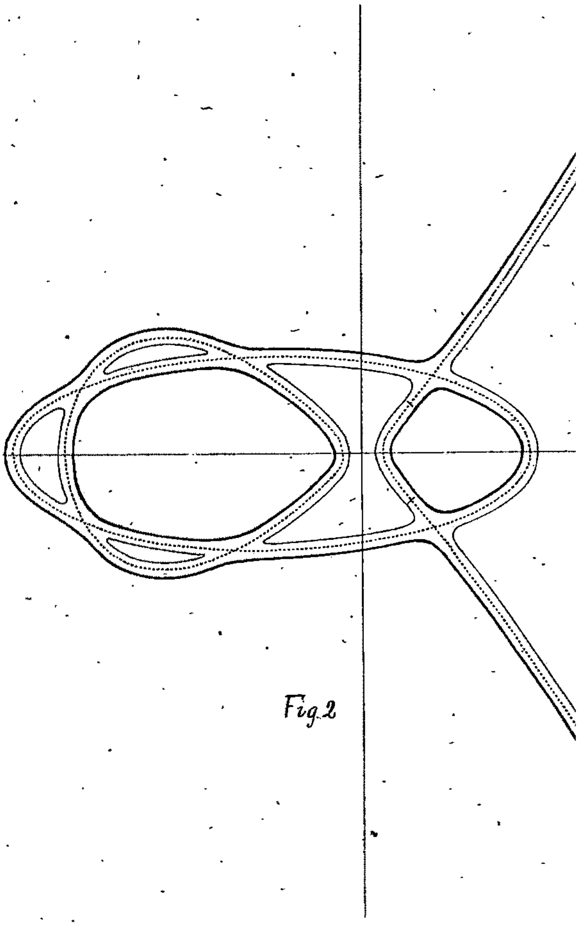


Fig. 2

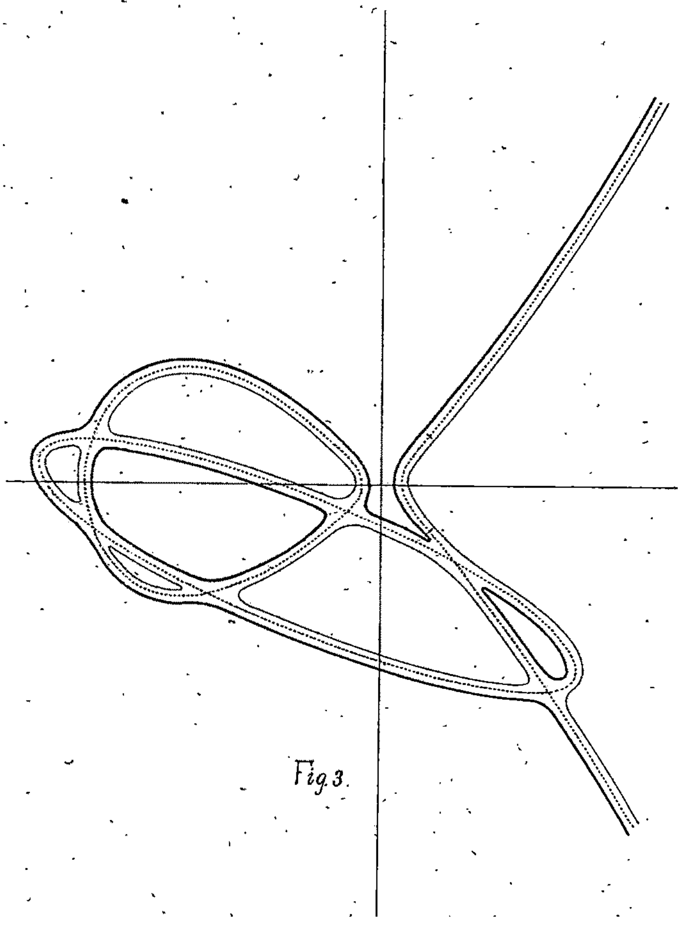


Fig. 3.

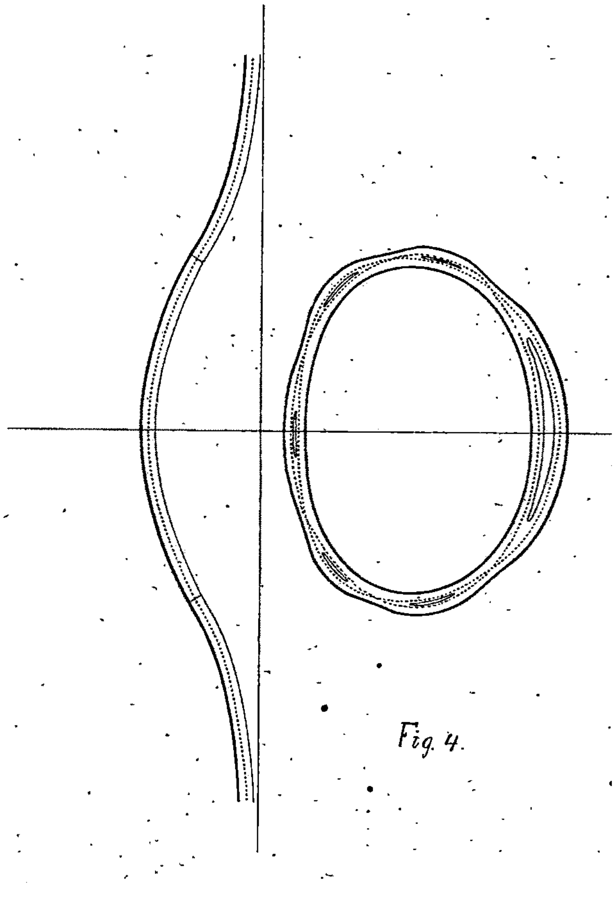


Fig. 4.

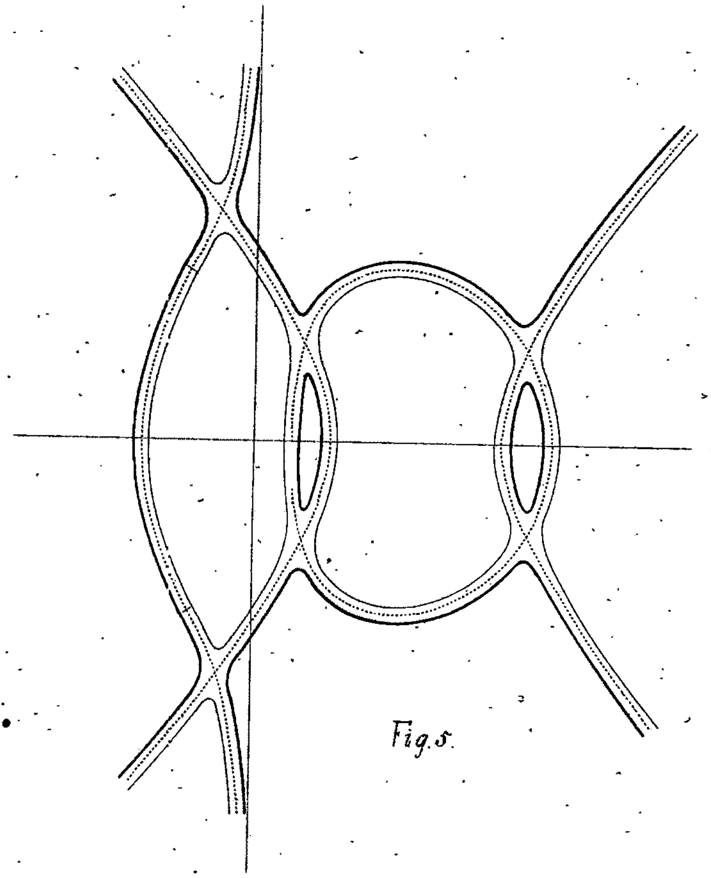


Fig. 5.

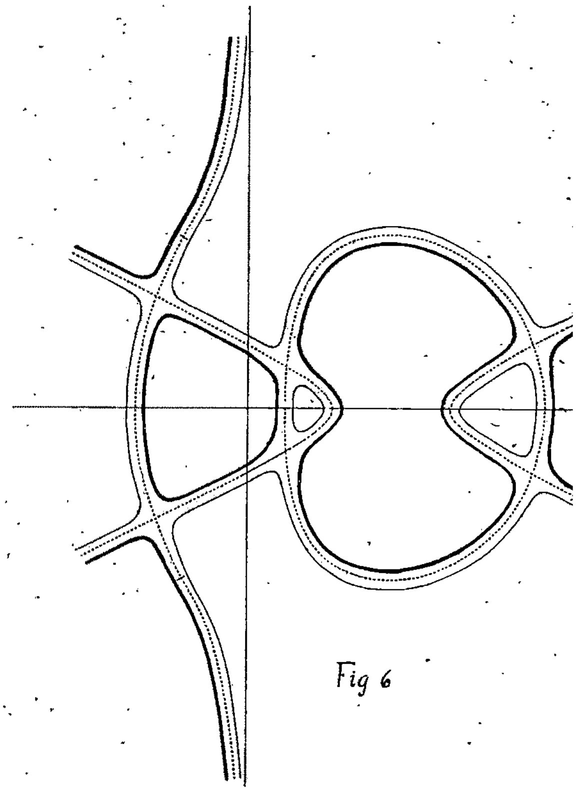
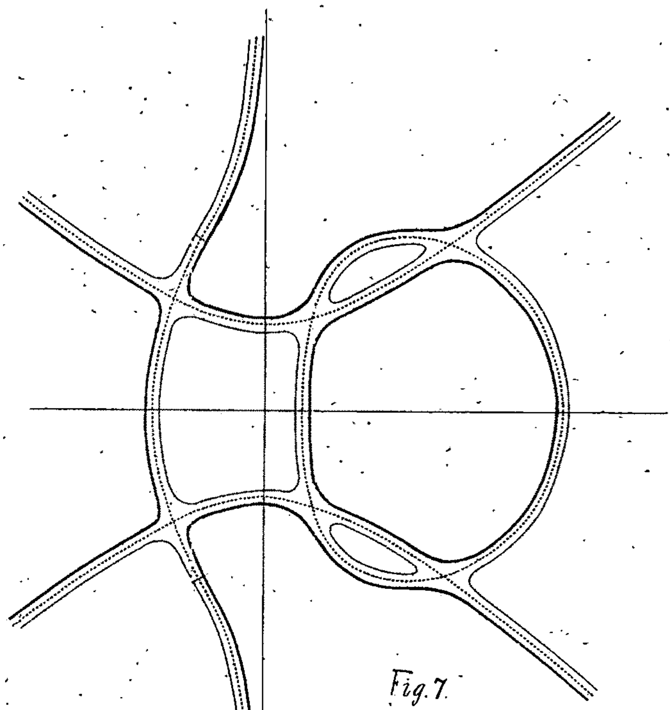


Fig 6



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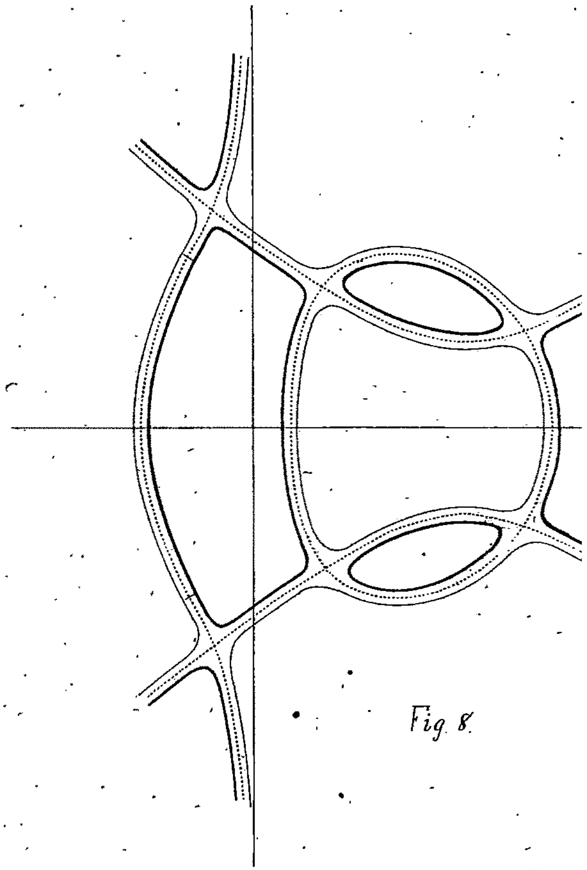

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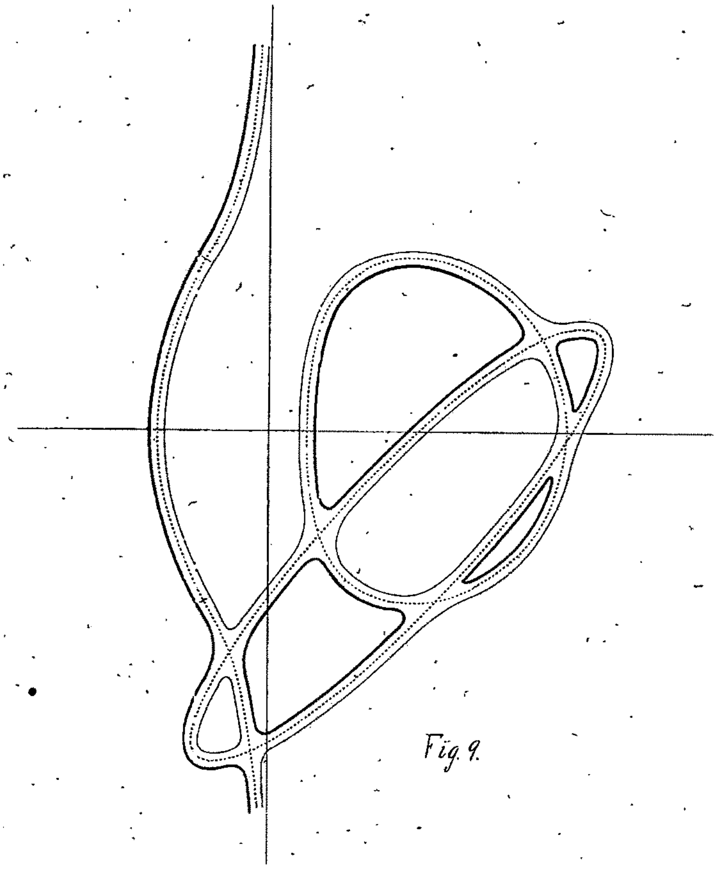
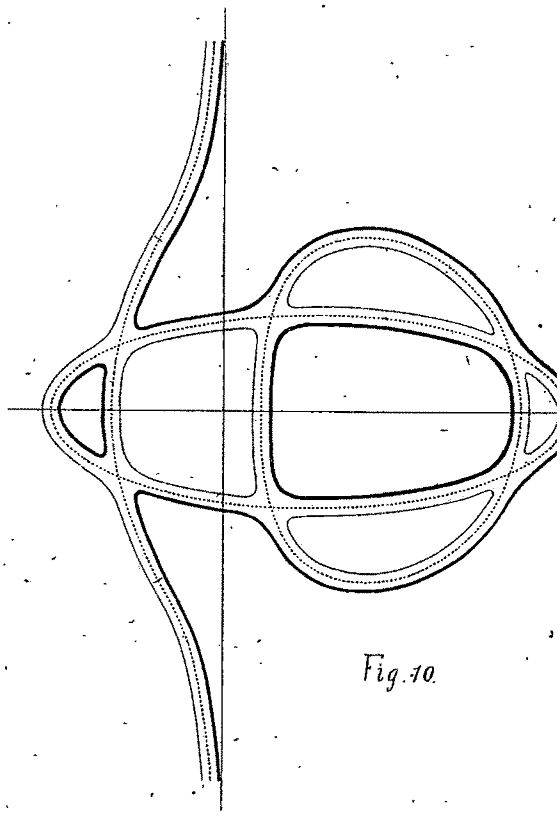


Fig. 9.



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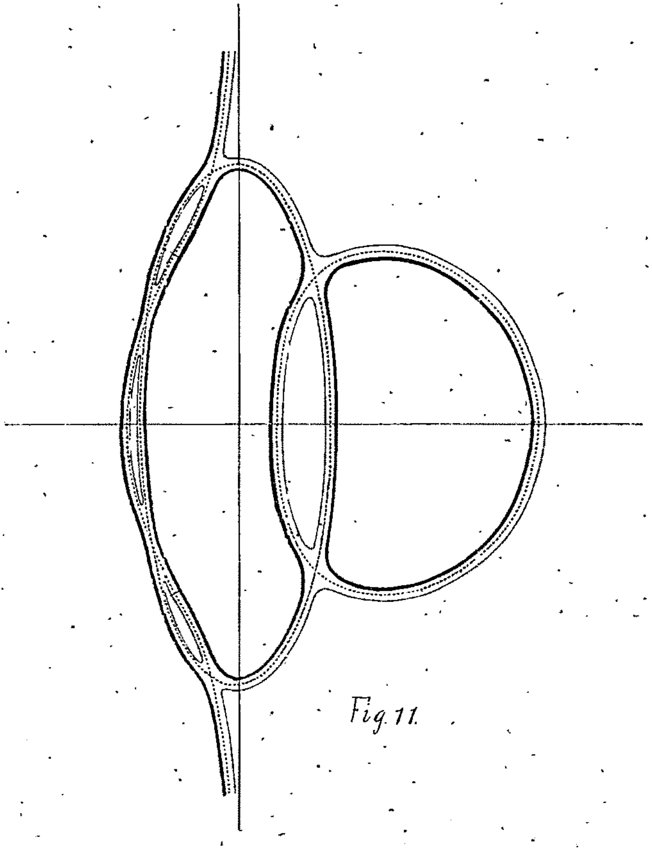

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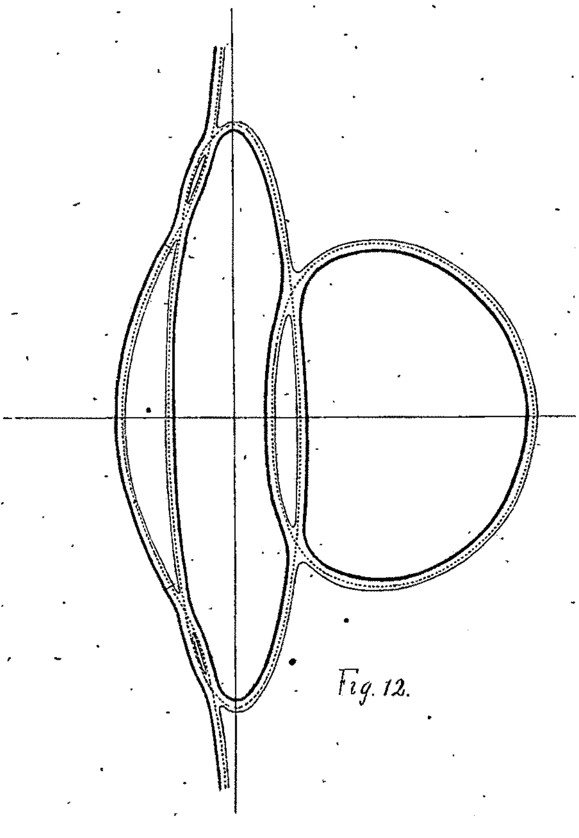

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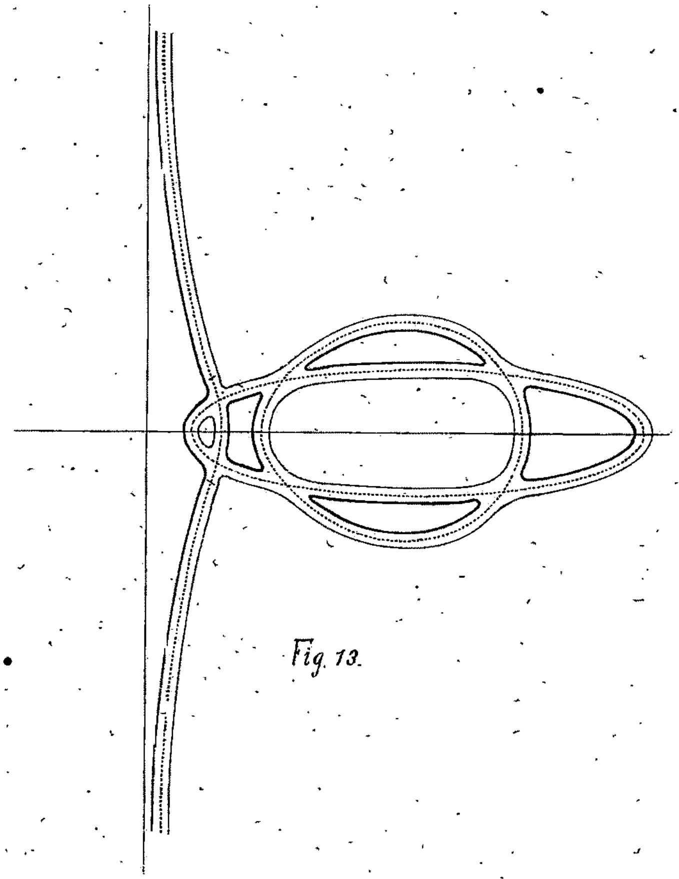
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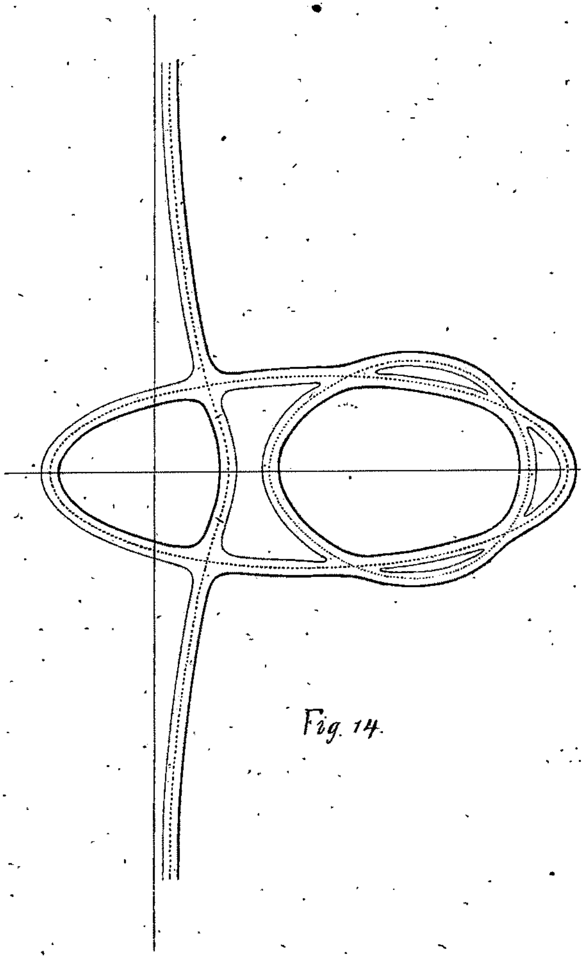
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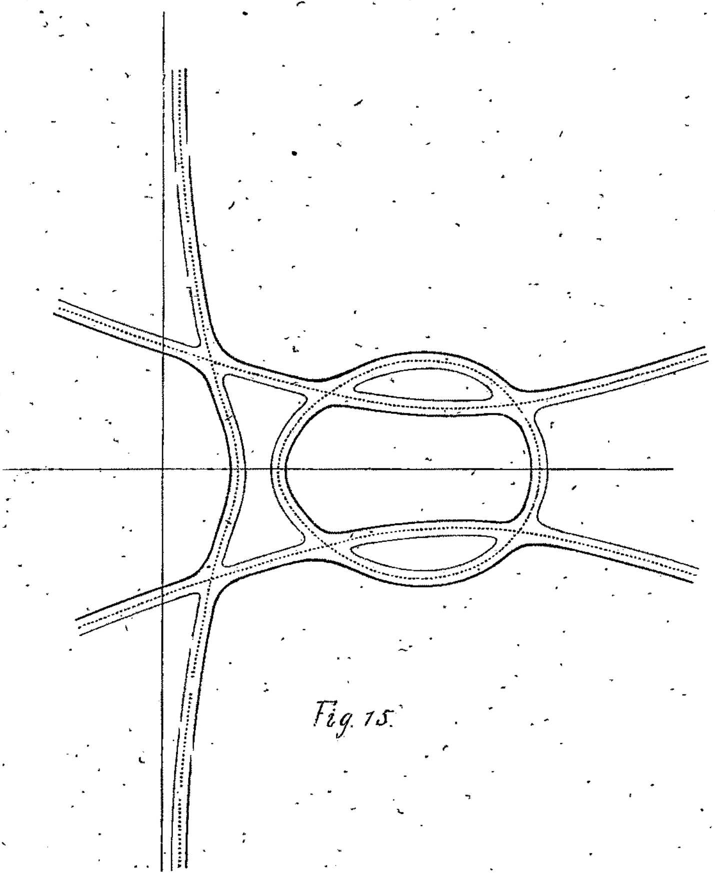
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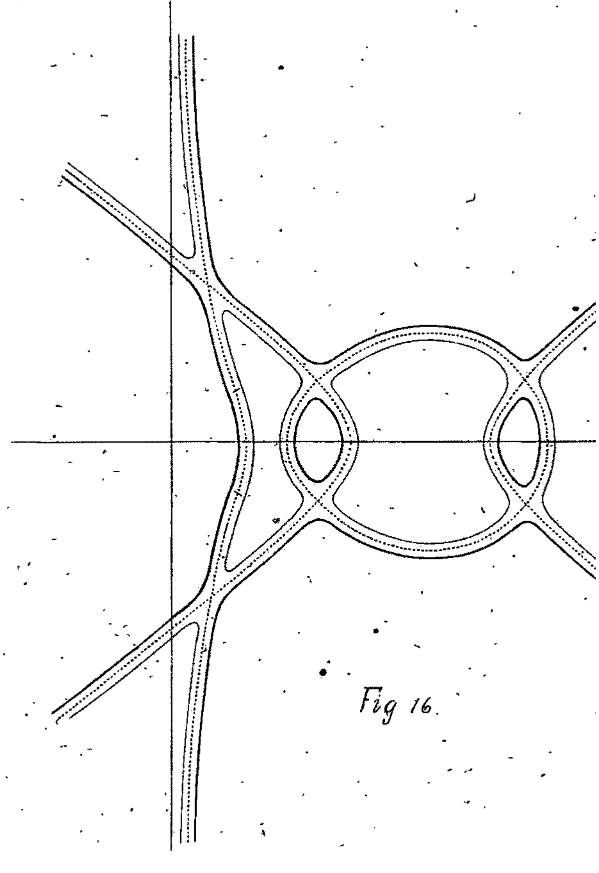
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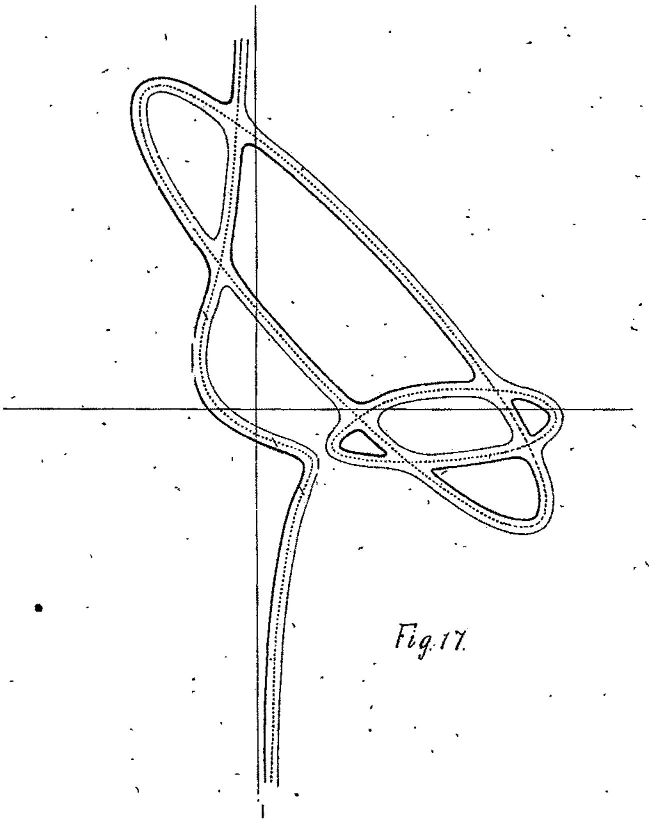

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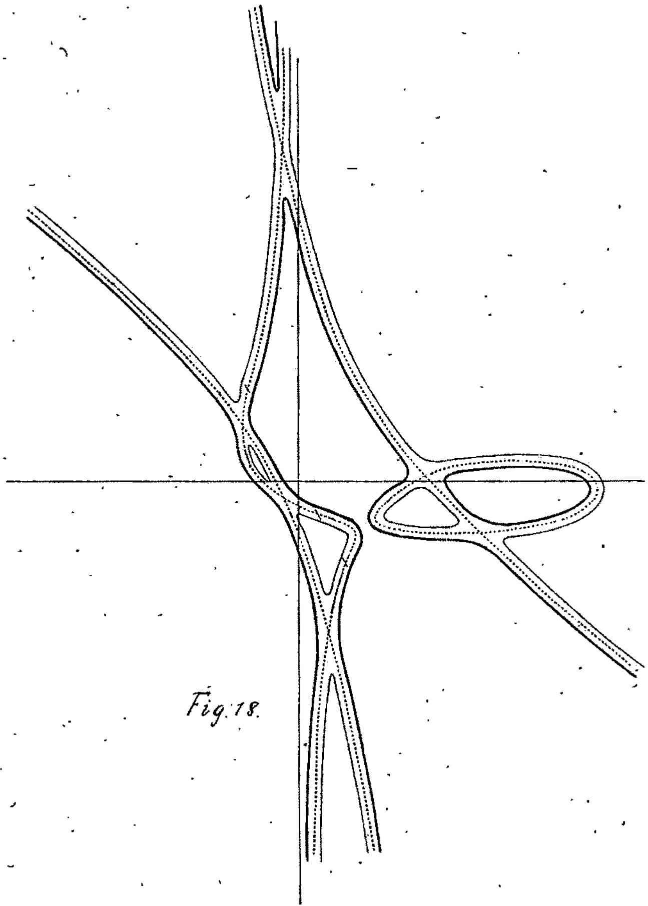

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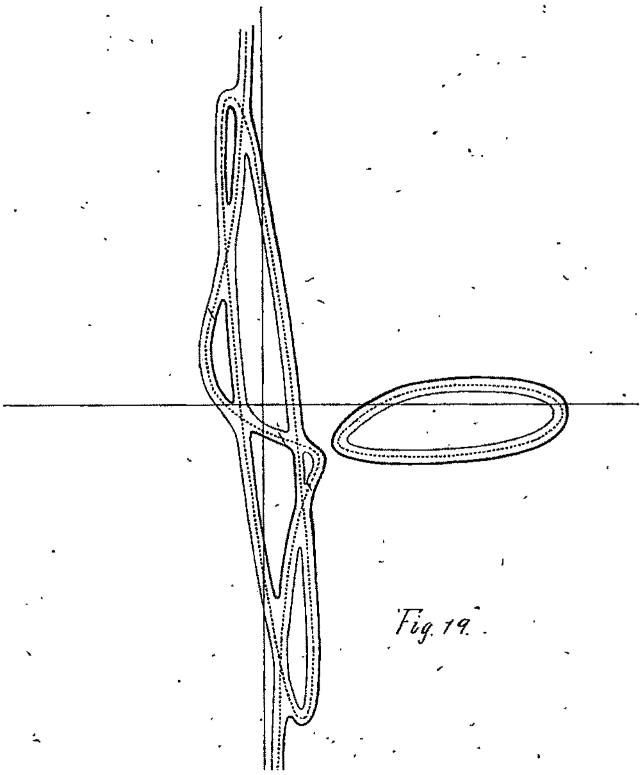

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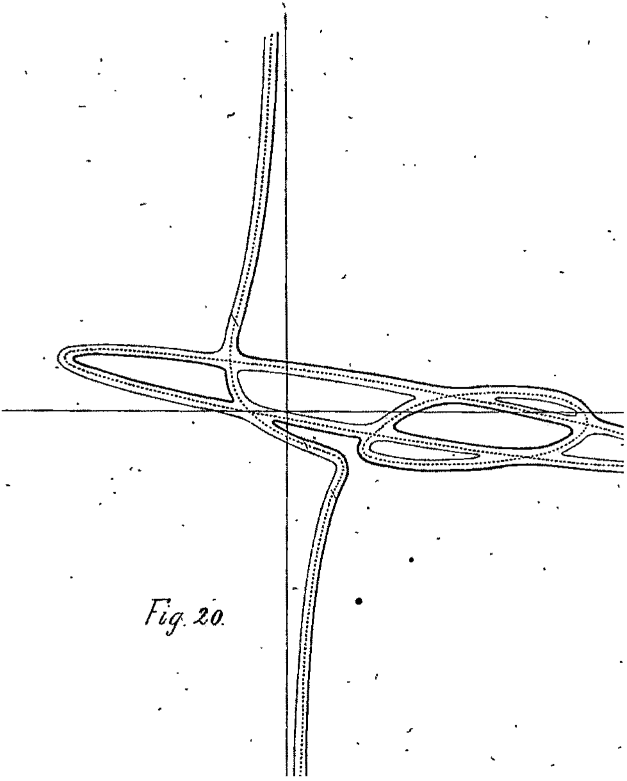

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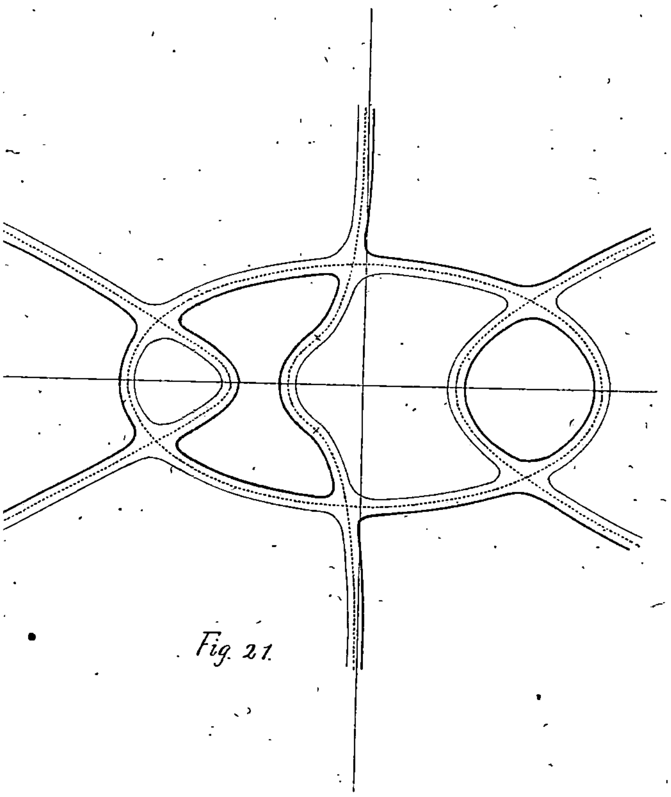

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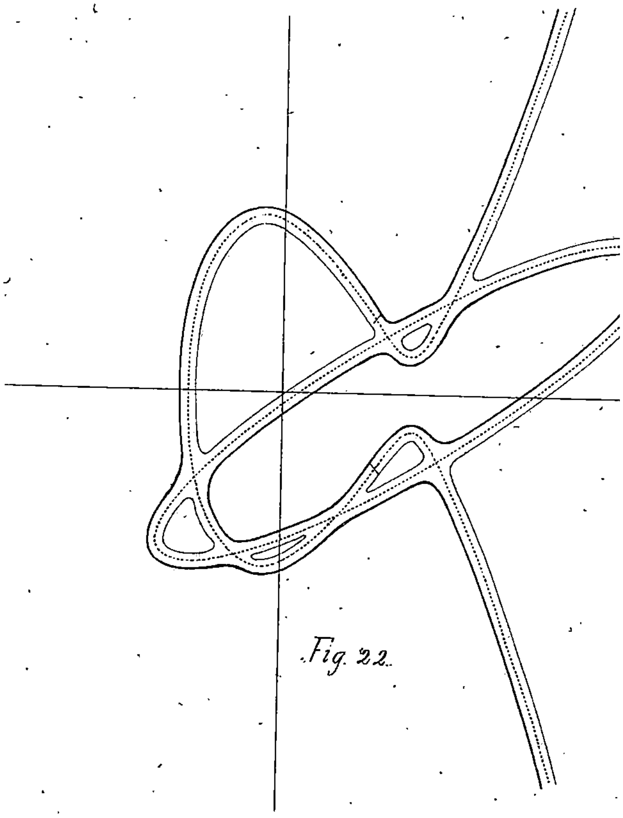

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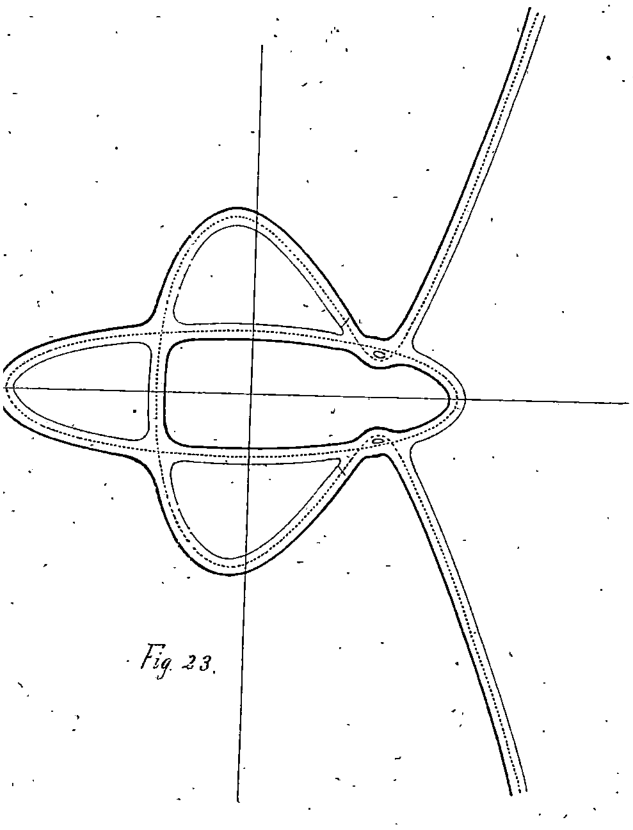

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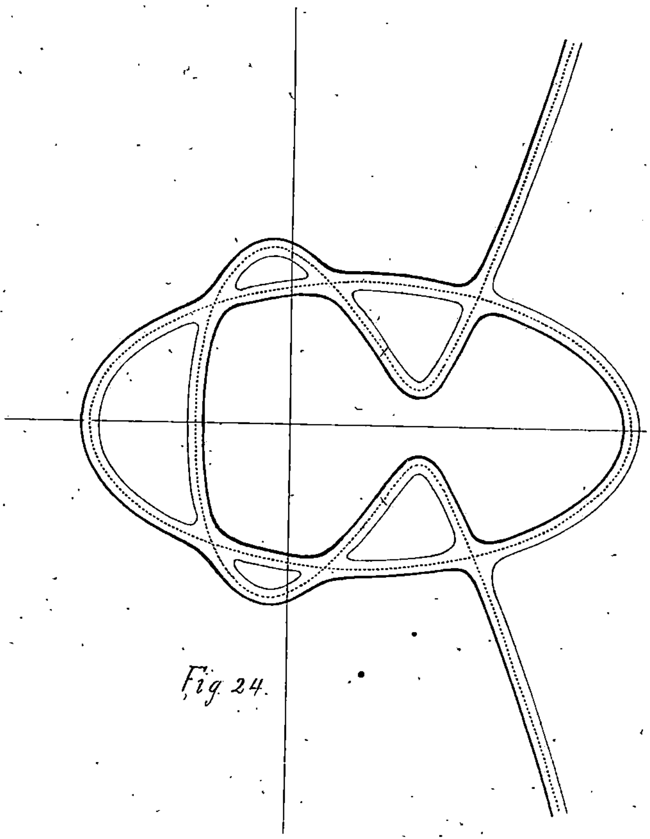

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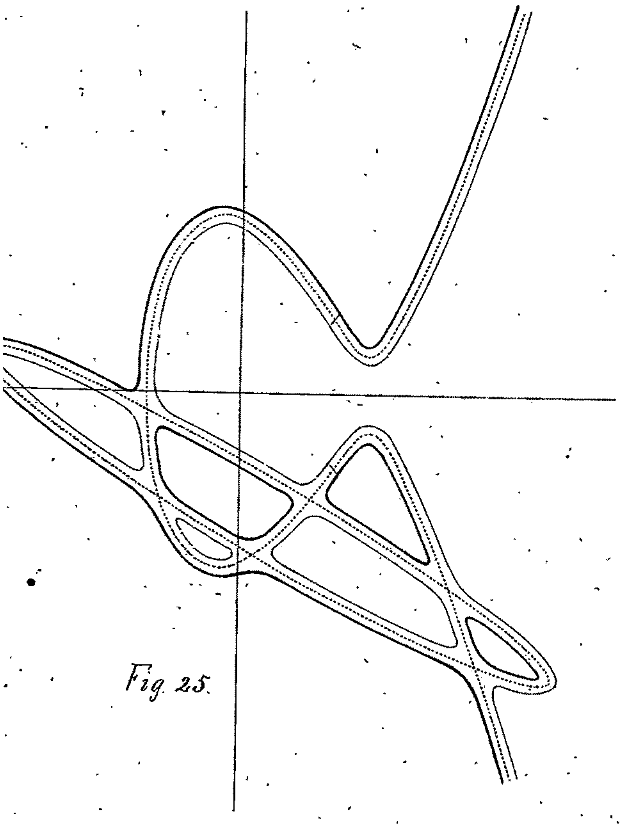

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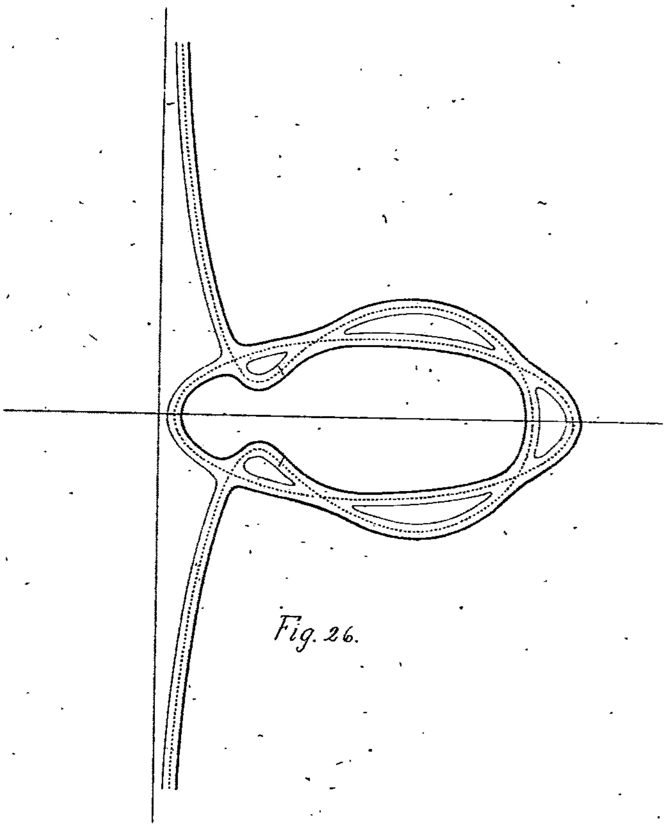

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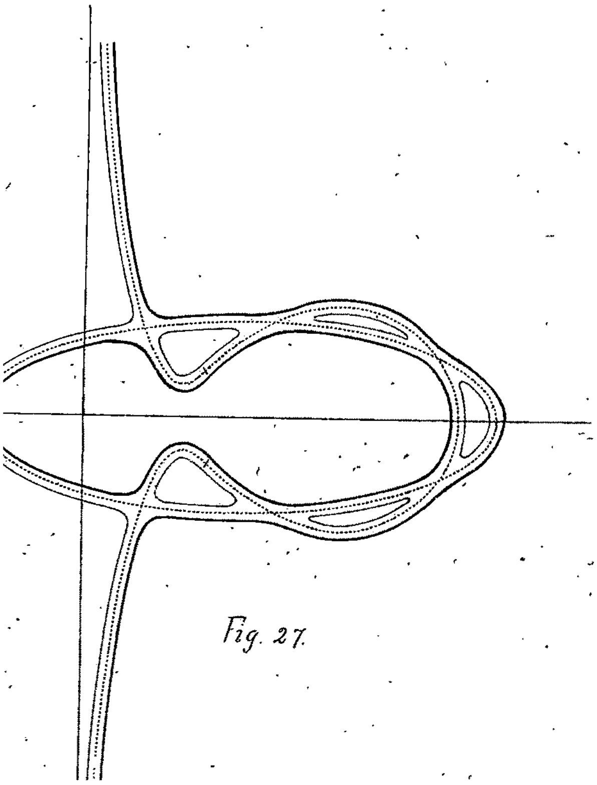

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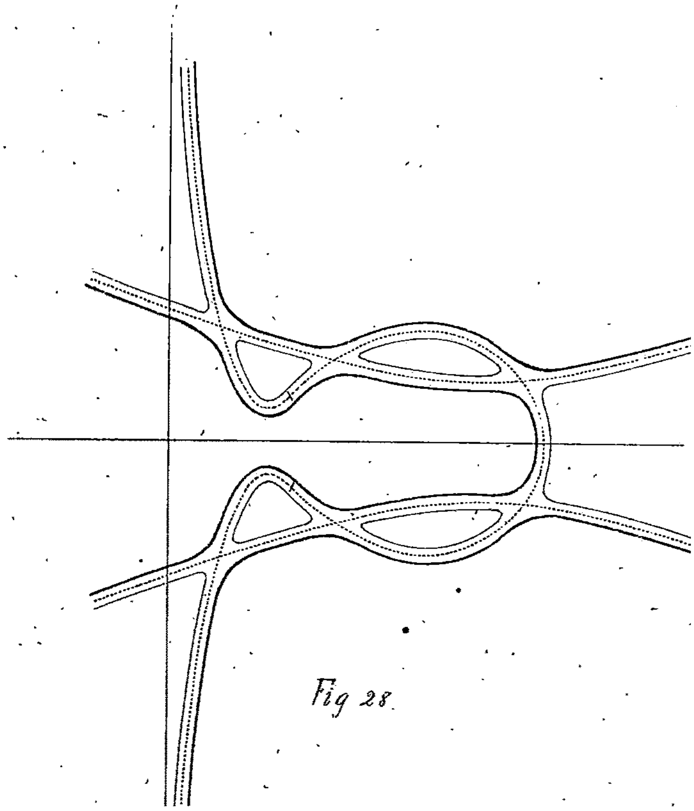

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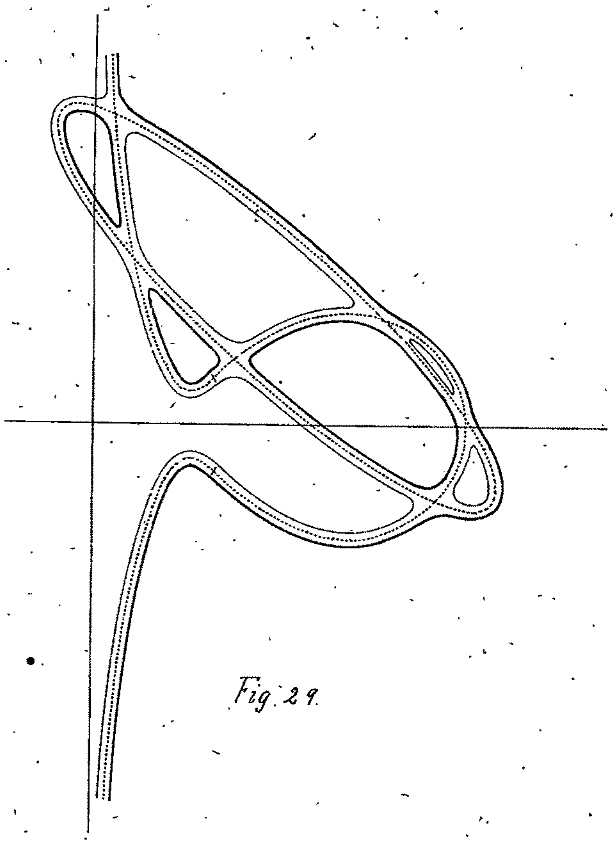

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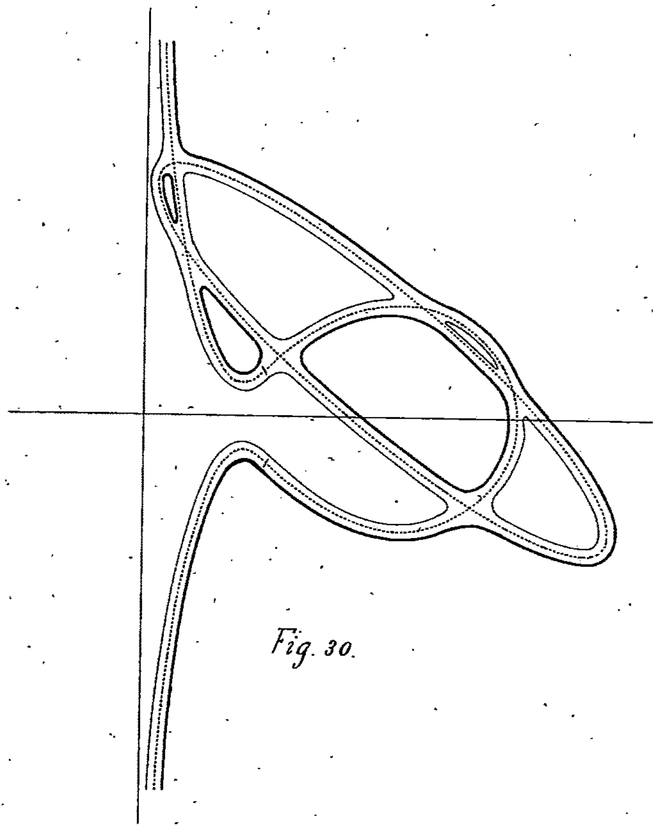

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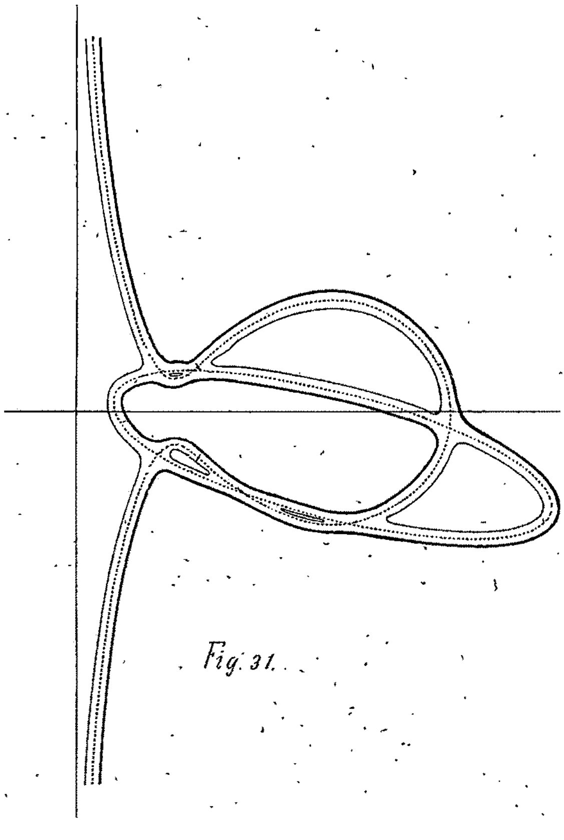

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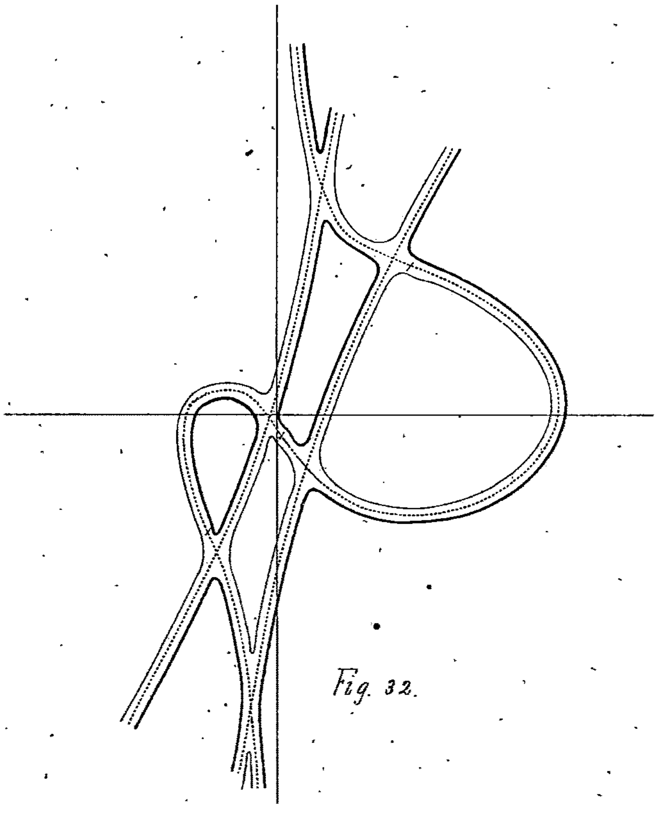

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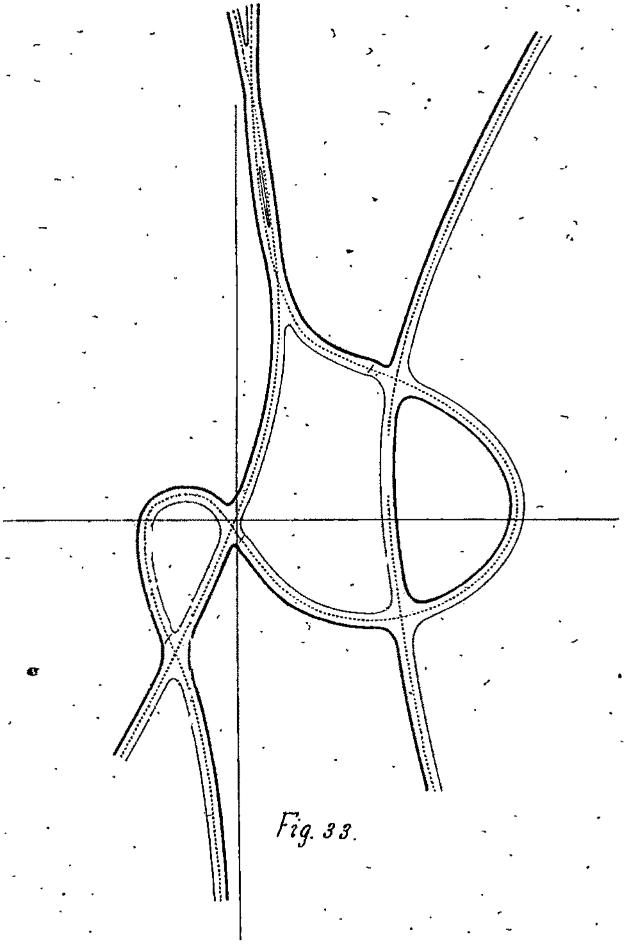

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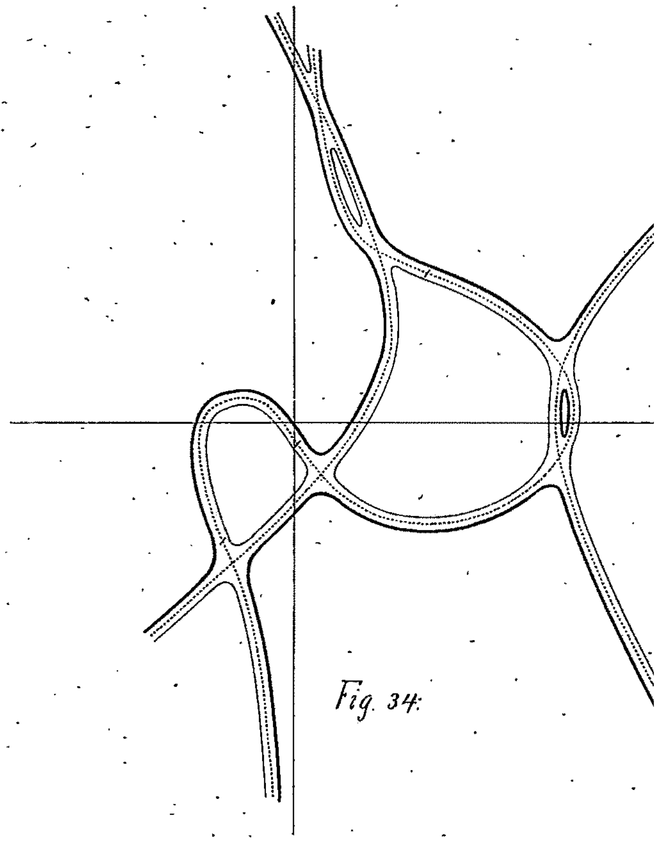

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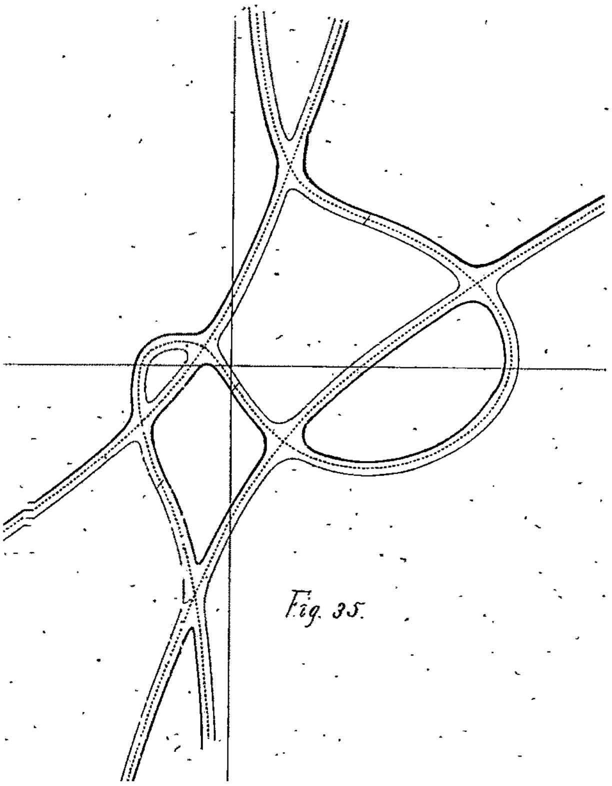

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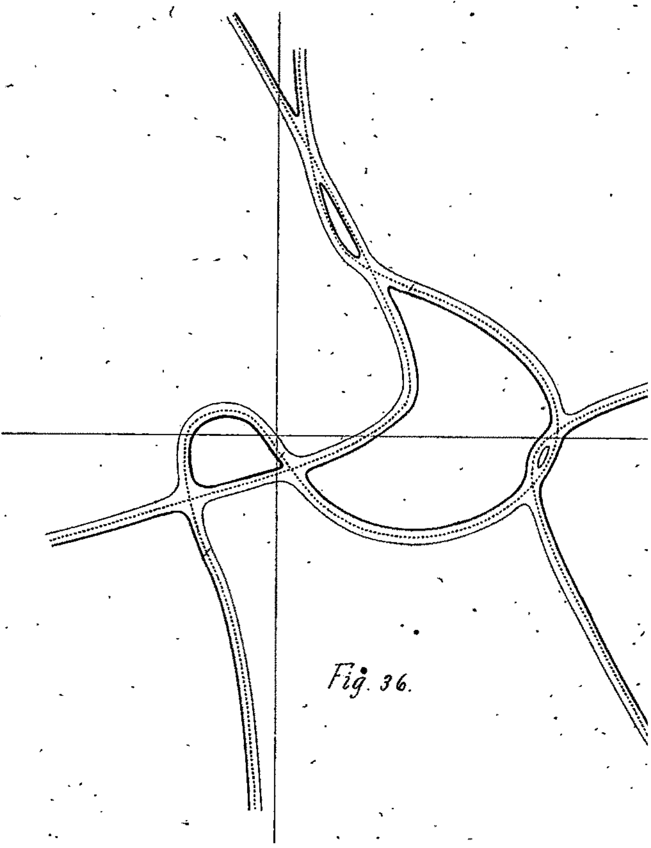

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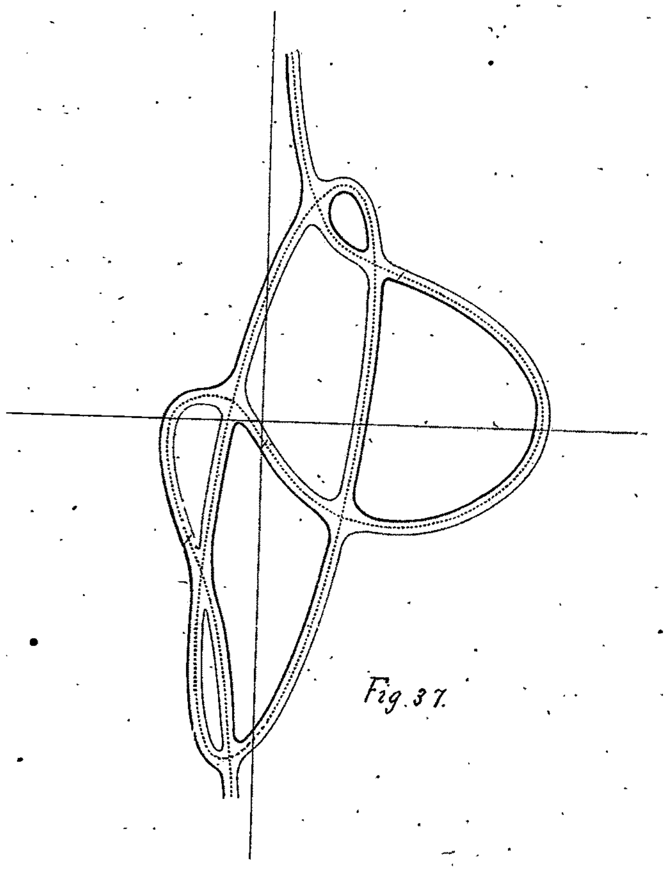

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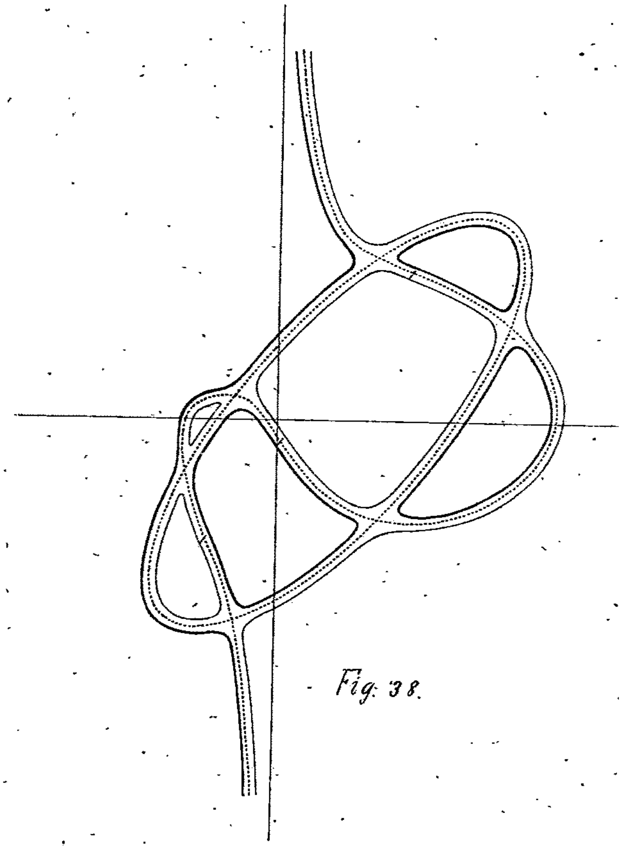

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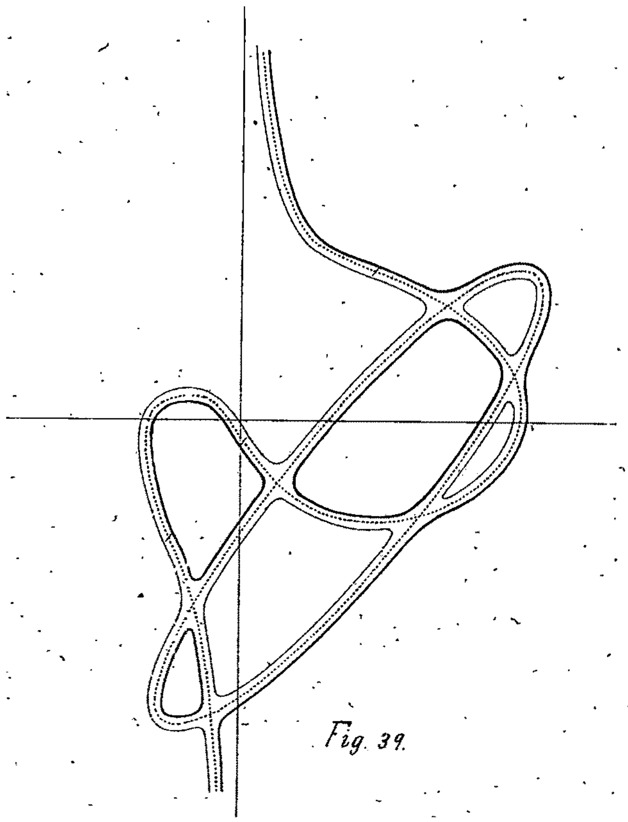

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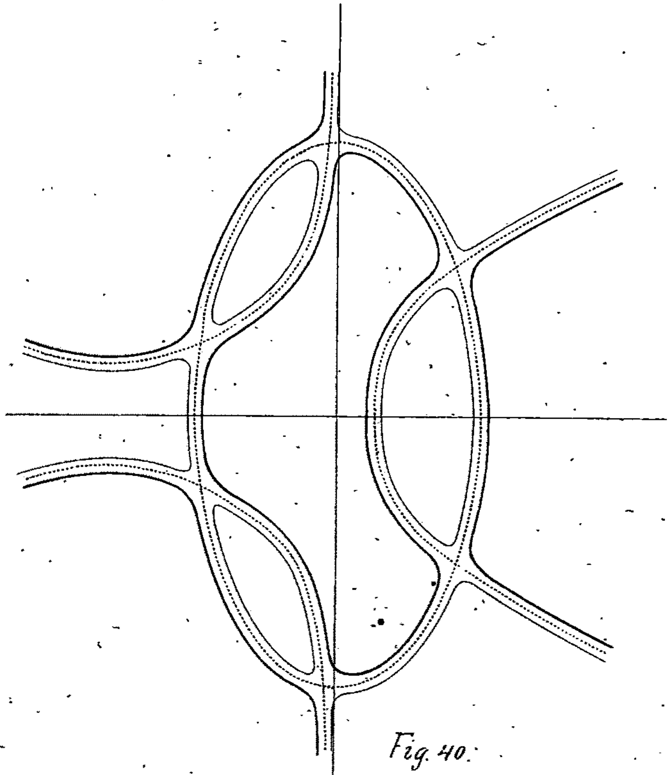

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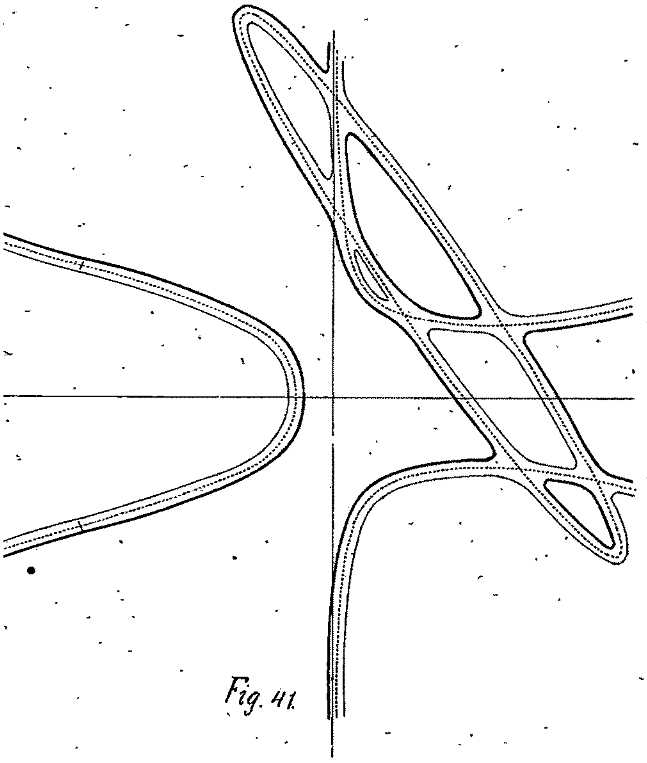

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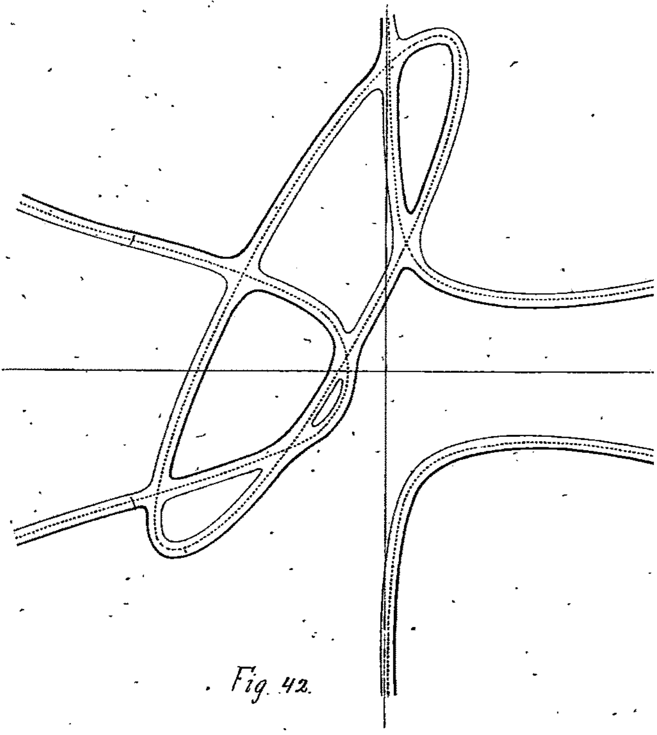

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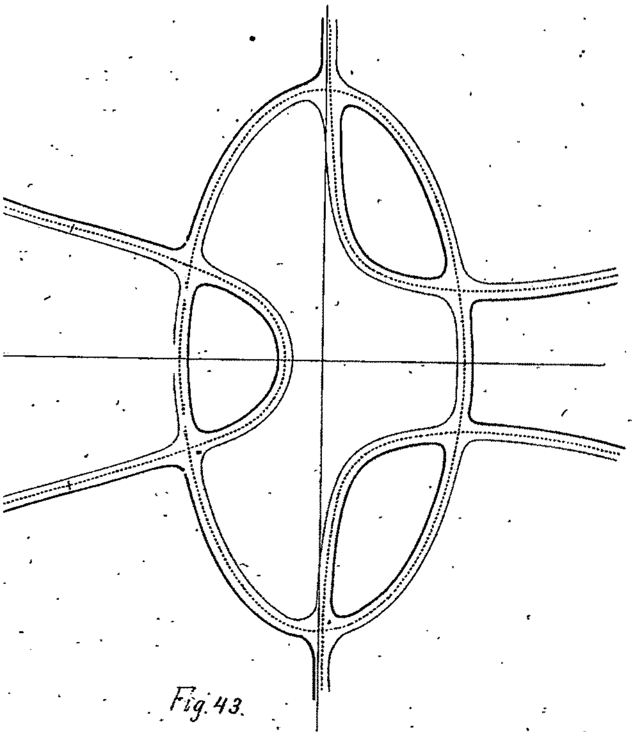

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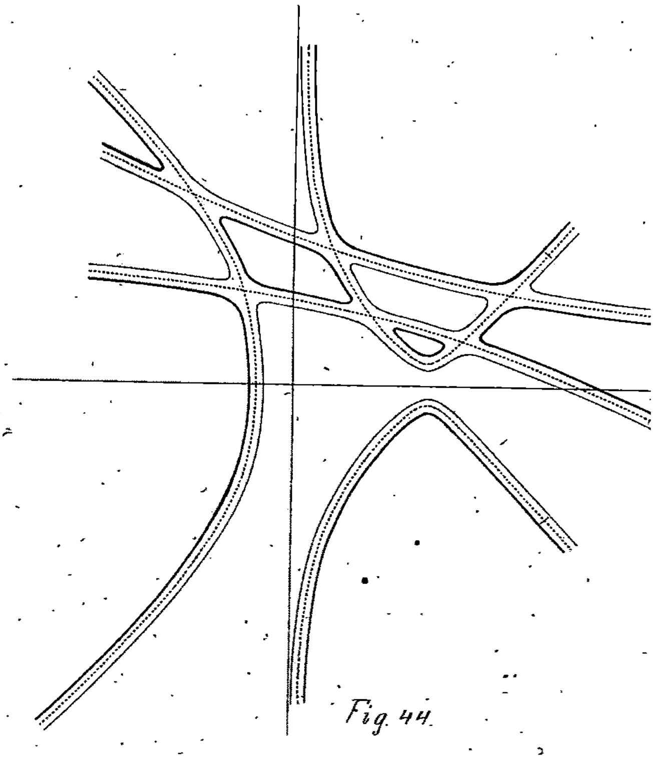

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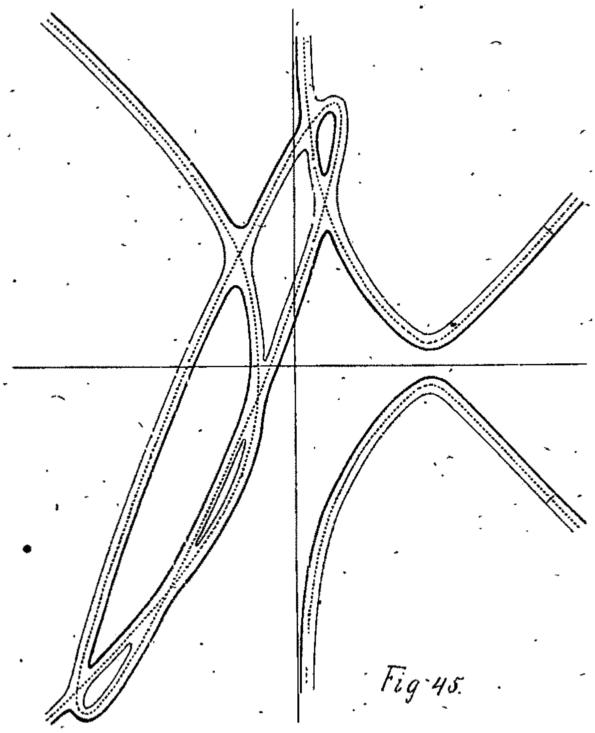
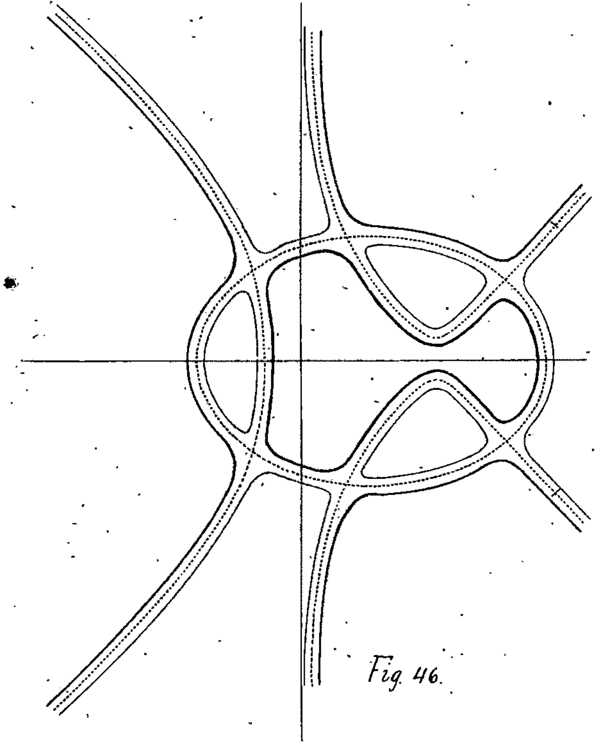


Fig 45.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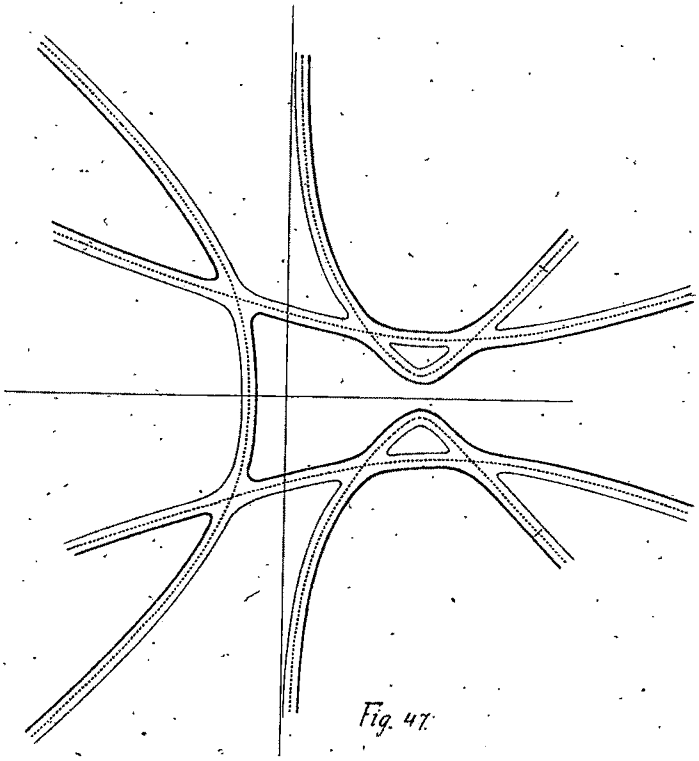
Fig. 47.

On Critic Centres.

By Frank Morley.

§1. The twelve lines each containing three of the nine inflexions of a cubic will intersect in twelve other points. For, taking any such line, through an inflexion on it three other lines can be drawn. We have thus ten of the lines; the other two contain the other six inflexions, and intersect in a point which is not an inflexion. From the twelve lines we get twelve points, which are clearly the critic centres or possible double points of all cubics having the same inflexions as the one considered.

For the canonical form

$$U \equiv x^3 + y^3 + z^3 + 6mxyz = 0,$$

the inflexions (I) are

$$\begin{array}{ccc} 0\ 1 - 1, & 0\ 1 - \omega, & 0\ 1 - \omega^2, \\ -1\ 0\ 1, & -\omega\ 0\ 1, & -\omega^2\ 0\ 1, \\ 1 - 1\ 0, & 1 - \omega\ 0, & 1 - \omega^2\ 0, \end{array}$$

where $1, \omega, \omega^2$ are cube roots of 1.

The twelve lines (L) are seen to be

$$\begin{array}{cccc} x, & x + y + z, & \omega x + y + z, & \omega^2 x + y + z, \\ y, & x + \omega y + \omega^2 z, & x + \omega y + z, & x + \omega^2 y + z, \\ z, & x + \omega^2 y + \omega z, & x + y + \omega z, & x + y + \omega^2 z, \end{array}$$

each column of three lines containing the nine points I.

Now the harmonic polar of a point of inflexion, say $0,\ 1,\ -1$, is $y - z$. For the polar conic $y^2 + 2mzx = z^2 + 2mxy$ breaks up into this line and the inflexional tangent $y + z - 2mx$. Thus we have corresponding to I the nine harmonic polars

$$\begin{array}{ccc} y - z, & y - \omega^2 z, & y - \omega z, \\ -x + z, & -\omega^2 x + z, & -\omega x + z, \\ x - y, & x - \omega^2 y, & x - \omega y, \end{array}$$

so that the line corresponding to any point is got by writing the coordinates as coefficients, interchanging ω and ω^2. The points corresponding to the lines L (*i. e.* the critic centres) are seen to be

$$\begin{array}{llll} 1\,0\,0, & 1\,1\,1, & \omega^2\,1\,1, & \omega\,1\,1, \\ 0\,1\,0, & 1\,\omega^2\,\omega, & 1\,\omega^2\,1, & 1\,\omega\,1, \\ 0\,0\,1, & 1\,\omega\,\omega^2, & 1\,1\,\omega^2, & 1\,1\,\omega, \end{array}$$

so that the point corresponding to any line is got by writing the coefficients as coordinates, interchanging ω and ω^2. From the symmetry the critic centres might be called the harmonic poles of the corresponding lines of inflexions. It is clear from the tables that the three harmonic polars of collinear inflexions meet in a point, the harmonic pole of the line of inflexions (an obvious property which I have not seen stated); that reciprocally the four harmonic poles of concurrent lines of inflexions lie on a line, the harmonic polar of the point (just as four lines L pass through each point I). Each column of three points forms the same triangle as the corresponding column of the lines L; on the sides are the nine inflexions, and through the angles pass the nine harmonic polars.

The points which have x as polar are given by $y^2 + 2mzx = 0$, $z^2 + 2mxy = 0$; the chords of intersection are

$$y^2+2mzx-(z^2+2mxy),\ y^2+2mzx-\omega(z^2+2mxy),\ y^2+2mzx-\omega^2(z^2+2mxy)=0;$$

hence the harmonic pole is one, the others are the triangle formed by the inflexional tangents and lie on the harmonic polars. The table of inflexional tangents is

$$\begin{array}{lll} -2mx+y+z, & -2mx+\omega^2 y+\omega z, & -2mx+\omega y+\omega^2 z, \\ x-2my+z, & \omega x-2my+\omega^2 z, & \omega^2 x-2my+\omega z, \\ x+y-2mz, & \omega^2 x+\omega y-2mz, & \omega x+\omega^2 y-2mz, \end{array}$$

and the polar of $1, 0, 0$ with regard to the triangle T formed by the first row is $x = 0$, since $\frac{dT}{dy} = 0$ and $\frac{dT}{dz} = 0$; so that the harmonic pole is the pole of the line L with regard to the three inflexional tangents. There are two lines L through a harmonic pole, and two such points on a line L; the polar conic of such a point (*e. g.* $x^2 + 2myz$) will touch the lines through it at the other poles on the lines.

To show the effect on a curve of passing through critic centres, we may say (what is easily verified) that a quartic through the harmonic poles inflects at each, the three tangents at points in a column meeting on the quartic.

§2. The polar conic of a point $x_1y_1z_1$ or O is

$$x_1x^2+y_1y^2+z_1z^2+2m(x_1yz+y_1zx+z_1xy)=0.$$

Eliminating m between this and U, we get for the locus of contacts of tangents from O,

$$Q\equiv 3xyz(x_1x^2+y_1y^2+z_1z^2)=(x^3+y^3+z^3)(x_1yz+y_1zx+z_1xy).$$

This quintic Q is seen to pass through O, the twelve harmonic poles, and the nine points I. Now the latter points are also inflexions on Q. For since

$$\frac{dQ}{dx}=3yz(x_1x^2+\ldots)+6x_1x^2yz-3x^2(x_1yz+\ldots)-(x^3+\ldots)(y_1z+z_1y),$$

the tangent at $0, 1, -1$ is

$$x(y_1+z_1)=x_1(y+z),$$

which passes through O. Eliminating x between this and Q,

$$\begin{aligned}&3(y_1+z_1)x_1yz(y+z)\{x_1^3(y+z)^2+(y_1+z_1)^2(y_1y^2+z_1z^2)\}\\ &=\{x_1^3(y+z)^3+(y_1+z_1)^3(y^3+z^3)\}\{x_1(y_1+z_1)yz+x_1(y+z)(y_1z+z_1y)\},\end{aligned}$$

or dividing by $x_1(y+z)$,

$$\begin{aligned}&3x_1^3(y_1+z_1)yz(y+z)^2+3yz(y_1+z_1)^3(y_1y^2+z_1z^2)\\ &-x_1^3(y_1+z_1)yz(y+z)^2-yz(y_1+z_1)^4(y^2-yz+z^2)\\ &-x_1^3(y_1z+z_1y)(y+z)^3-(y_1+z_1)^3(y^3+z^3)(y_1z+z_1y)=0;\end{aligned}$$

the terms of which $y+z$ is not an apparent factor may be written

$$3yz(y_1+z_1)^3\{y_1y^2+z_1z^2+yz(y_1+z_1)\}-yz(y_1+z_1)^4(y^2+z^2+2yz),$$

or $\quad 3yz(y_1+z_1)^3(y+z)(y_1y+z_1z)-yz(y_1+z_1)^4(y+z)^2.$

Dividing by $y+z$,

$$\begin{aligned}2x_1^3(y_1+z_1)yz(y+z)-x_1^3(y_1z+z_1y)(y+z)^2-(y_1+z_1)^3(y^2-yz+z^2)(y_1z+z_1y)&\\ +3yz(y_1+z_1)^3(y_1y+z_1z)-yz(y_1+z_1)^4(y+z)=0&.\end{aligned}$$

The terms of which $y+z$ is not an apparent factor may be written

$$(y_1+z_1)^3\{3yz(y_1y+z_1z+y_1z+z_1y)-(y^2+z^2+2yz)(y_1z+z_1y)\};$$

or $\quad (y_1+z_1)^3\{3yz(y+z)(y_1+z_1)-(y+z)^2(y_1z+z_1y)\}.$

Dividing by $y+z$,

$$\begin{aligned}x_1^3\{2yz(y_1+z_1)-(y+z)(y_1z+z_1y)\}+(y_1+z_1)^3\{3yz(y_1+z_1)&\\ -(y+z)(y_1z+z_1y)\}-yz(y_1+z_1)^4=0&,\end{aligned}$$

or $\quad 2yz(y_1+z_1)-(y+z)(y_1z+z_1y)=0,$

or $\quad (y-z)(y_1z-z_1y)=0.$

Thus the lines joining O to I are inflexional tangents to Q, and meet Q again on the corresponding harmonic polars.

From the equation of Q it passes through the intersections of

$$x_1x^2 + y_1y^2 + z_1z^2 = 0,\quad x_1yz + y_1zx + z_1xy = 0,$$

which are the four fixed points on the polar conic of O with regard to U. If we write X, Y for the special cubics $x^3 + y^3 + z^3$, $3xyz$; X', Y' for the first polars of O with regard to them; X'', Y'' for the second polars; X''', Y''' for their values at O, then the quintic and its polars may be seen, by successive operations with $x_1\frac{d}{dx} + y_1\frac{d}{dy} + z_1\frac{d}{dz}$, to be

$$YX' = XY',$$
$$YX'' = XY'',$$
$$Y'X'' + YX''' = X'Y'' + XY''',$$

or, since there is the identity, as may be seen by ordinary algebra,

$$Y'X'' - YX''' \equiv X'Y'' - XY''',$$

the cubic is
$$YX''' = XY''' \quad\text{or}\quad Y'X'' = X'Y'';$$

differentiating either, the conic and line are

$$Y'X''' = X'Y''',$$
$$Y''X''' = X''Y'''.$$

From the first form for the cubic it goes through I and O, and from the second it meets the polar conic at the points $X' = 0$, $Y' = 0$, and the polar line at the point $X'' = 0$, $Y'' = 0$, which is the fixed point on the polar line of O with regard to U. The polar quartic passes through the same point (P suppose), so that OP is a double tangent to the quartic.

We have now to show that Q is the general form of a quintic through the 21 points. A quintic about the triangle of reference is

$$x^4(b_1y + c_1z) + y^4(c_2z + a_2x) + z^4(a_3x + b_3y) + x^3(e_1y^2 + f_1z^2) + y^3(f_2z^2 + d_2x^2)$$
$$+ z^3(d_3x^2 + e_3y^2) + xyz(ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy) = 0.$$

Putting $x = 0$,

$$yz(c_2y^3 + b_3z^3 + f_2y^2z + e_3yz^2)$$

must
$$\equiv yz(y^3 + z^3),$$
$$\therefore\ d_2,\ d_3,\ e_3,\ e_1,\ f_1,\ f_2 = 0,$$

and
$$c_2 = b_3 = \alpha,\ a_3 = c_1 = \beta,\ b_1 = a_2 = \gamma,$$

so that the curve is

$$\left.\begin{aligned} &\alpha yz(y^3 + z^3) + \beta zx(z^3 + x^3) + \gamma xy(x^3 + y^3) \\ &+ xyz(ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy) \end{aligned}\right\} = 0.$$

Substituting $1\,1\,1$, $\omega\,1\,1$, $\omega^2\,1\,1$,

$$\begin{aligned} 2(\alpha+\beta+\gamma)+a+b+c+2(f+g+h)&=0,\\ 2(\alpha+\beta\omega+\gamma\omega)+a+b\omega+c\omega+2(f\omega+g\omega^2+h\omega^2)&=0,\\ 2(\alpha+\beta\omega^2+\gamma\omega^2)+a+b\omega^2+c\omega^2+2(f\omega^2+g\omega+h\omega)&=0,\end{aligned}$$

$$\therefore\ 2\alpha+a=0,$$

or

$$a=-2\alpha,\ b=-2\beta,\ c=-2\gamma,$$

$$\therefore\ f\omega^2+g+h=0,$$

$$f\omega+g+h=0,$$

$$f+g+h=0,$$

$$\therefore\ g+h=0,\ h+f=0,\ f+g=0,\ \text{or}\ f,\,g,\,h=0.$$

Hence the general form is

$$\alpha yz(y^3+z^3-2x^3)+\beta zx(z^3+x^3-2y^3)+\gamma xy(x^3+y^3-2z^3)=0,$$

which is clearly equivalent to Q.

We have not used the points $1\,\omega\,\omega^2$, $1\,\omega^2\,\omega$, and need not have used $1\,1\,1$; so that a quintic through I and nine of the harmonic poles goes through the other three. Moreover, it goes through them without being further defined, and still contains two parameters. There is here a remarkable anomaly, for, while twenty points determine a quintic, we have a quintic through twenty-one points with two disposable constants.

For the other intersections of two such quintics we have

$$\frac{x_1x^2+y_1y^2+z_1z^2}{x_1yz+y_1zx+z_1xy}=\frac{x^3+y^3+z^3}{3xyz}=\alpha\ \text{say},$$

or

$$x_1(x^2-\alpha yz)+\ldots\ldots=0$$

and

$$x_2(x^2-\alpha yz)+\ldots\ldots=0,$$

$$\therefore\ \frac{x^2-\alpha yz}{y_1z_2-z_1y_2}=\ldots\ldots$$

or since $x^3+y^3+z^3-3\alpha xyz=0$,

$$\begin{vmatrix} x & y & z\\ x_1 & y_1 & z_1\\ x_2 & y_2 & z_2 \end{vmatrix}=0;$$

that is to say, the remaining four intersections of two quintics whose parametric points are O, O' lie on the line OO'. These will be fixed for quintics $Q+\lambda Q'$, for their parametric points will be on OO', but not for quintics whose parametric points are off OO'. Hence though quintics through nineteen fixed points pass in general through six other fixed points, in this case we have quintics through twenty-one points whose remaining intersections may lie on any line not through the points. (If the parametric point is one of the twenty-one points, it is easily seen that the quintic splits up.) The proof that the other points of intersection are fixed is evaded by the existence of the two parameters (see Salmon's Plane Curves, p. 18). As was seen above, the twelve harmonic poles count only as nine conditions in fixing the quintic.

§3. It is clear that most of §2 is true in general. Let U, V be two n-ics, and let ξ denote $a\dfrac{d}{dx}+b\dfrac{d}{dy}+c\dfrac{d}{dz}$. Eliminating λ from

$$U+\lambda V=0,$$
$$\xi U+\lambda\xi V=0,$$

we get

$$Q\equiv V\xi U-U\xi V=0,$$

a $(2n-1)$-ic through the n^2 intersections, and also the $3(n-1)^2$ critic centres, for which

$$\frac{U_1}{V_1}=\frac{U_2}{V_2}=\frac{U_3}{V_3}=\frac{\xi U}{\xi V}=\frac{U}{V}.$$

The first polar of a, b, c or O is

$$\xi Q\equiv V\xi^2U-U\xi^2V,$$

which passes through $U=0$, $V=0$. Hence the tangents to Q at these points go through O. The points of contact of the other tangents from O lie on

$$\xi V.\xi^2U-\xi U.\xi^2V=0,$$

which touches Q at O, the common tangent being

$$\xi^{n-1}V.\xi^nU-=0,$$

ξ^nU being a constant.

To get the νth polar we may expand $\xi^\nu(V\xi U-U\xi V)$, remembering that Leibnitz's theorem applies to ξ, or use the more convenient form

$$\xi^{\nu+1}(VU)-2\xi^\nu(\xi V.U)-=0,$$

where now in each expansion we stop half way, *i. e.* keep only the terms in which the power of ξ acting on the first factor $<$ that acting on the second. The equivalence of the methods is clear if we write down the expansions. The polar cubic has a singular property; its equation becomes

$$n\xi^{n-2}V.\xi^{n-1}U+3(n-2)\xi^{n-3}V.\xi^nU-=0,$$

where, as above, the — precedes terms got by interchanging U and V. Now identically

$$\xi^{n-2}V.\xi^{n-1}U - \xi^{n-3}V.\xi^{n}U - = 0,$$

for if we operate with ξ, the left vanishes, and it also vanishes if $x, y, z = 0$. Hence the polar cubic is either

$$\xi^{n-2}V.\xi^{n-1}U - = 0,$$

or

$$\xi^{n-3}V.\xi^{n}U \quad - = 0,$$

and passes through the intersections of

$$\xi^{n-1}U,\ \xi^{n-1}V;\ \xi^{n-2}U,\ \xi^{n-2}V;\ \xi^{n-3}U,\ \xi^{n-3}V.$$

Thus the fixed point P_1 on the polar line of O with regard to $U + \lambda V$, the four fixed points P_2 on the polar conic, and the nine fixed points P_3 on the polar cubic lie on a cubic through O; OP_1 is a tangent at O to the cubic, and OP_2 are tangents at P_2.

Let Q' be the curve obtained from $a'b'c'$ or O'. Then for the intersections

$$U/V = \xi U/\xi V = \xi' U/\xi' V.$$

A possible set of points is

$$U = 0,\ \xi U = 0,\ \xi' U = 0,$$

or

$$\begin{aligned} xU_1 + yU_2 + zU_3 &= 0, \\ aU_1 + bU_2 + cU_3 &= 0, \\ a'U_1 + b'U_2 + c'U_3 &= 0, \end{aligned}$$

whence

$$\begin{vmatrix} x & y & z \\ a & b & c \\ a' & b' & c' \end{vmatrix} = 0.$$

The remaining $(2n-1)^2 - n^2 - 3(n-1)^2$ or $2(n-1)$ intersections $\therefore$ lie on OO'. These are fixed for $Q + \lambda Q'$, but not for curves whose parametric points lie off OO'. Since $(2n-1)(n+1)$ conditions fix a curve Q, and besides the $n^2 + 3(n-1)^2$ we require two more, it appears likely that all such curves through all but $n^2 + 3(n-1)^2 + 2 - (2n-1)(n+1)$, or $(n-2)(2n-3)$, of the critic centres pass through the rest. I have not noticed a similar difficulty for curves of even degree; but, at least for the case of the critic centres, the accepted theory appears to require limitation.

HAVERFORD COLLEGE, PA., October, 1887.

NOTE.—Professor Cayley has kindly cleared up the difficulty stated above; it comes under Bacharach's addition to Cayley's original theorem. See Cayley, "On the Intersection of Curves," Math. Annalen, Vol. XXX (1887), pp. 85–90; Bacharach, *ib.* XXVI (1885), pp. 275–299. On the point in question, Professor Cayley writes: "In fact, four points may be represented by $P_1 = 0$, $Q_4 = 0$; we have the quintics $A_4P_1 + B_1Q_4 = 0$, $A_4'P_1 + B_1'Q_4 = 0$, meeting in these four points and in twenty-one other points; the quintic $A_4B_1' - A_4'B_1 = 0$ passes through these twenty-one points but not through the four points—or generally we have

$$\alpha (A_4P_1 + B_1Q_4) + \alpha' (A_4'P_1 + B_1'Q_4) + \gamma (A_4B_1' - A_4'B_1) = 0,$$

a quintic through the twenty-one points but not through the four points." Thus, when we have shown that two $(2n - 1)$-ics through the n^2 intersections and $3(n-1)^2$ critic centres of two n-ics meet again in points on a right line, we do not expect other $(2n-1)$-ics through the $n^2 + 3(n-1)^2$ points to share the other $2(n-1)$ points.

The Expression of Syzygies among Perpetuants by means of Partitions.

By Captain P. A. MacMahon, R. A.

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This paper is entirely based upon the partition expression of seminvariants; the previous papers on the subject are, in the order of their publication,

The Author.	Seminvariants and Symmetric Functions (*A. J. M.*, Vol. 6, p. 131).
Cayley.	A Memoir on Seminvariants (*A. J. M.*, Vol. 7, p. 1).
The Author.	On Perpetuants (*A. J. M.*, Vol. 7, p. 26).
	A second paper on Perpetuants (*A. J. M.*, Vol. 7, p. 259).
	A Memoir on Seminvariants (*A. J. M.*, Vol. 8, p. 1).
Hammond.	On Perpetuants with Applications to the Theory of Finite Quantics (*A. J. M.*, Vol. 8, p. 104).

§1.

Assume the binary quantic to be

$$(a_0, a_1, a_2, \dots)(x_0, x_1)^p,$$

wherein it will be observed the variables are written x_0, x_1, and moreover are supposed hypothetically of weights 0 and 1; any coefficient in the quantic has thus the same weight as the combination of variables which it affects.

Write any two covariants of this quantic

$$u \equiv (A_0, A_1, A_2, \dots)(x_0, x_1)^s,$$
$$v \equiv (B_0, B_1, B_2, \dots)(x_0, x_1)^\sigma,$$

and represent the differential operation

$$\frac{d^{p+q}}{dx_0^p\, dx_1^q},$$

by $\xi_0^p \xi_1^q$ or by $\eta_0^p \eta_1^q$,

according as it takes effect upon u or upon v; the operation of Ueberschiebung of order $\varkappa$ is then symbolically $(\xi_0\eta_1 - \xi_1\eta_0)^\kappa$,

and noting that

$$\xi_0^p \xi_1^q \eta_0^{p'} \eta_1^{q'} = \frac{s!\,\sigma!}{(s-p-q)!\,(\sigma-p'-q')!}(A_q, A_{q+1}, A_{q+2}, \dots)(x_0, x_1)^{s-p-q}$$
$$(B_{q'}, B_{q'+1}, B_{q'+2}, \dots)(x_0, x_1)^{\sigma-p'-q'},$$

we find at once

$$\frac{(s-\varkappa)!\,(\sigma-\varkappa)!}{s!\qquad\sigma!}(\xi_0\eta_1 - \xi_1\eta_0)^\kappa$$
$$= (A_0, A_1, A_2, \dots)(x_0, x_1)^{s-\kappa}(B_\kappa, B_{\kappa+1}, B_{\kappa+2} \dots)(x_0, x_1)^{\sigma-\kappa}$$
$$- \varkappa(A_1, A_2, A_3, \dots)(x_0, x_1)^{s-\kappa}(B_{\kappa-1}, B_\kappa, B_{\kappa+1} \dots)(x_0, x_1)^{\sigma-\kappa}$$
$$+ \frac{\varkappa.\varkappa-1}{2!}(A_2, A_3, A_4, \dots)(x_0, x_1)^{s-\kappa}(B_{\kappa-2}, B_{\kappa-1}, B_\kappa, \dots)(x_0, x_1)^{\sigma-\kappa}$$
$$- \dots;$$

or if we write $(A_q, A_{q+1}, A_{q+2}, \dots)(x_0, x_1)^{s-\kappa} = S_q$,

$$(B_{q'}, B_{q'+1}, B_{q'+2}, \dots)(x_0, x_1)^{\sigma-\kappa} = \Sigma_{q'},$$

we may exhibit the result in the symbolic form

$$\frac{(s-\varkappa)!\,(\sigma-\varkappa)!}{s!\qquad\sigma!}(\xi_0\eta_1 - \xi_1\eta_0)^\kappa = (\Sigma - S)^\kappa,$$

wherein $(\Sigma - S)^\kappa$ is to be expanded in the form

$$\Sigma^\kappa S^0 - \varkappa\Sigma^{\kappa-1}S' + \dots,$$

and then each power changed into a suffix.

Hence, neglecting a numerical multiplier, the source of the covariant

$$(\xi_0\eta_1 - \xi_1\eta_0)^\kappa$$

is in a symbolic form $(A - B)^\kappa$,

or at length

$$A_\kappa B_0 - \kappa A_{\kappa-1}B_1 + \frac{\kappa(\kappa-1)}{2!}A_{\kappa-2}B_2 - \ldots + (-)^\kappa A_0 B_\kappa.$$

Giving κ the values $0, 1, 2, 3, \ldots,$

we write down the series of seminvariants:

$$\begin{aligned}&A_0B_0,\\ &A_1B_0 - A_0B_1,\\ &A_2B_0 - 2A_1B_1 + A_0B_2,\\ &A_3B_0 - 3A_2B_1 + 3A_1B_2 - A_0B_3,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\\ &\ldots\ldots\ldots\ldots\ldots\ldots\\ &\ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}$$

each source is derived from the next preceding by the operation of

$$A_1\partial_{A_0} + A_2\partial_{A_1} + \ldots - B_1\partial_{B_0} - B_2\partial_{B_1} - \ldots,$$

and further, each is annihilated by

$$A_0\partial_{A_1} + 2A_1\partial_{A_2} + \ldots + B_0\partial_{B_1} + 2B_1\partial_{B_2} + \ldots$$

§2.

Write the reversor

$$\rho a_1\partial_{a_0} + (\rho - 1)a_2\partial_{a_1} + (\rho - 2)a_3\partial_{a_2} + \ldots + a_\rho\partial_{a_{\rho-1}}$$

in the form

$$\rho\left\{a_1\partial_{a_0} + \left(1 - \frac{1}{\rho}\right)a_2\partial_{a_1} + \left(1 - \frac{2}{\rho}\right)a_3\partial_{a_2} + \ldots + \frac{1}{\rho}a_\rho\partial_{a_{\rho-1}}\right\}$$

$= \rho\delta$ suppose; then

$$\begin{aligned}u &= A_0x_0^s + \rho\delta A_0x_0^{s-1}x_1 + \frac{1}{2!}\rho^2\delta^2A_0x_0^{s-2}x_1^2 + \ldots\\ &= x_0^s e^{\frac{x_1}{x_0}\rho\delta}A_0 \qquad (s \equiv \rho = \infty);\end{aligned}$$

but if the quantic be actually supposed of unlimited order, and A_0 be of degree-weight (α, w), then

$$\delta = a_1\partial_{a_0} + a_2\partial_{a_1} + a_3\partial_{a_2} + \ldots,$$

$$\rho = \frac{s}{\alpha} \text{ in the limit,}$$

and the asymptotic form of the covariant u is

$$\left(A_0,\ \frac{\delta A_0}{\alpha},\ \frac{\delta^2 A_0}{\alpha^2},\ \ldots\ldots\right)(x_0,\ x_1)^s,$$

or symbolically,

$$x_0^s\left(1+\frac{x_1}{x_0}\,\frac{\delta}{\alpha}\right)^s A_0.$$

The process of Ueberschiebung now gives the series

$$\begin{array}{l} A_0B_0, \\ \beta\delta A_0.B_0-\alpha A_0\delta B_0, \\ \beta^2\delta^2A_0.B_0-2\alpha\beta\delta A_0.\delta B_0+\alpha^2A_0.\delta^2B_0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}$$

and in general,

$$\beta^\kappa\delta^\kappa A_0.B_0-\kappa\beta^{\kappa-1}\alpha\delta^{\kappa-1}A_0\delta B_0+\frac{\kappa.\kappa-1}{2!}\beta^{\kappa-2}\alpha^2\delta^{\kappa-2}A_0\delta^2B_0-\ldots\ldots,$$

or with the proviso that $(\delta A_0)^0=\delta^0A_0=A_0$; $\delta^jA_0=(\delta A_0)^j$, this is, in symbolic form,

$$(\beta\delta A_0-\alpha\delta B_0)^\kappa.$$

The result just obtained is seen to be identical with that given by M. Perrin, and partially also by M. Maurice d'Ocagne, in recent numbers of the Comptes Rendus of the Academy of Sciences in Paris.*

§3.

Transformation of the Foregoing Results.

Consider the quantic with derived coefficients

$$(a_0,\ 1!\ a_1,\ 2!\ a_2,\ \ldots\ldots)(x_0,\ x_1)^\rho;$$

the sources become transformed into symmetric functions of the roots of the equation

$$a_0x^n-a_1x^{n-1}+a_2x^{n-2}-\ldots\ldots=0\quad(n=\infty)$$

in which no root occurs raised to the power unity; (N. B.—These have been elsewhere called non-unitary symmetric functions);

the reversor

$$\rho a_1\partial_{a_0}+(\rho-1)\,a_2\partial_{a_1}+(\rho-2)\,a_3\partial_{a_2}+\ldots\ldots$$

becomes

$$\begin{array}{l}\rho\,(a_1\partial_{a_0}+2a_2\partial_{a_1}+3a_3\partial_{a_2}+\ldots\ldots) \\ -2\,(a_2\partial_{a_1}+3a_3\partial_{a_2}+6a_4\partial_{a_3}+\ldots\ldots),\end{array}$$

---

* The connection of the result with the process of Ueberschiebung, and the fact that it is merely its expression quâ sources of quantics of unlimited order, was not mentioned by either M. Perrin or M. d'Ocagne.

wherein the numerical coefficients in the last portion of the operator are the successive triangular numbers.

Writing this reversor $\rho\delta_1 - 2\delta_2$,

it is necessary to separately examine δ_1 and δ_2. Put

$$d_\lambda \equiv a_0\partial_{a_\lambda} + a_1\partial_{a_{\lambda+1}} + a_2\partial_{a_{\lambda+2}} + \ldots.,$$

so that

$$\partial_{a_\lambda} = h_0\frac{d_\lambda}{a_0} - h_1\frac{d_{\lambda+1}}{a_0^2} + h_2\frac{d_{\lambda+2}}{a_0^3} - \ldots.,$$

wherein h_w denotes the sum of all the homogeneous products of the roots of the equation, of weight w; thus,

$$h_w = \Sigma(-)^{w+\alpha_1+\alpha_2+\ldots.}\frac{(\alpha_1+\alpha_2+\ldots.)!}{\alpha_1!\ \alpha_2!\ldots.}a_0^{\alpha_0}a_1^{\alpha_1}a_2^{\alpha_2}\ldots.,\qquad(\Sigma s\alpha_s = w);$$

whence

$$\begin{aligned}\delta_1 &= a_1\partial_{a_0} + 2a_2\partial_{a1} + 3a_3\partial_{a2} + 4a_4\partial_{a_3} + \ldots.\\
&= a_1\left(h_0\frac{d_0}{a_0} - h_1\frac{d_1}{a_0^2} + h_2\frac{d_2}{a_0^3} - \ldots.\right)\\
&+ 2a_2\left(h_0\frac{d_1}{a_0} - h_1\frac{d_2}{a_0^2} + h_2\frac{d_3}{a_0^3} - \ldots.\right)\\
&+ 3a_3\left(h_0\frac{d_2}{a_0} - h_1\frac{d_3}{a_0^2} + h_2\frac{d_4}{a_0^3} - \ldots.\right)\\
&+ \ldots\ldots\ldots\ldots\ldots\ldots\\
&= \frac{a_1h_0}{a_0}d_0 - \frac{a_1h_1 - 2a_2a_0h_0}{a_0^2}d_1 + \frac{a_1h_2 - 2a_2a_0h_1 + 3a_3a_0^2h_0}{a_0^3}d_2 - \ldots.\\
&= \sum_{\lambda=0}^{\lambda=\infty}(-)^\lambda\frac{a_1h_\lambda - 2a_2a_0h_{\lambda-1} + 3a_3a_0^2h_{\lambda-2} - \ldots. + (-)^{s+1}Sa_sa_0^{s-1}h_{\lambda-s+1} + \ldots.}{a_0^{\lambda+1}}d_\lambda\\
&= \sum_{\lambda=0}^{\lambda=\infty}(-)^\lambda S_{\lambda+1}d_\lambda \text{ by a known formula.}\end{aligned}$$

S_r denoting the sum of the r^{th} powers of the roots.

$$\therefore\ \delta_1 = S_1d_0 - S_2d_1 + S_3d_2 - \ldots.$$

Similarly

$$\begin{aligned}\delta_2 &= a_2\partial_{a_1} + 3a_3\partial_{a_2} + 6a_4\partial_{a_3} + \ldots.\\
&= a_2\left(h_0\frac{d_1}{a_0} - h_1\frac{d_2}{a_0^2} + h_2\frac{d_3}{a_0^3} - \ldots.\right)\\
&+ 3a_3\left(h_0\frac{d_2}{a_0} - h_1\frac{d_3}{a_0^2} + h_2\frac{d_4}{a_0^3} - \ldots.\right)\\
&+ 6a_4\left(h_0\frac{d_3}{a_0} - h_1\frac{d_4}{a_0^2} + h_2\frac{d_5}{a_0^3} - \ldots.\right)\\
&+ \ldots.\end{aligned}$$

$$= \frac{a_2h_0}{a_0}d_1 - \frac{a_2h_1 - 3a_3a_0h_0}{a_0^2}d_2 + \frac{a_2h_2 - 3a_3a_0h_1 + 6a_4a_0^2h_0}{a_0^3}d_3 - \ldots.$$

$$= \sum_{\lambda=1}^{\lambda=\infty}(-)^{\lambda+1}a_0^{-\lambda}$$

$$\left\{a_2h_{\lambda-1} - 3a_3a_0h_{\lambda-2} + 6a_4a_0^2h_{\lambda-3} + \ldots. + (-)^{\lambda+1}\frac{1}{2}\lambda(\lambda+1)a\quad {}_1a_0^{\lambda-1}h_0\right\}d_\lambda$$

$$= \sum_{\lambda=1}^{\lambda=\infty}(-)^{\lambda+1}S_{\lambda+1}^2d_\lambda \text{ by a known formula.}$$

$S_{\lambda+1}^2$ denoting the sum of all those symmetric functions of weight $\lambda + 1$ which involve two and only two roots in their expression; thus,

$$S_{\lambda+1}^2 = \Sigma\Sigma\alpha^p\beta^q\text{*}$$

where

$$p + q = \lambda + 1,$$
$$p > 0,\ q > 0.$$

We thus have

$$\delta_2 = S_2^2d_1 - S_3^2d_2 + S_4^2d_3 - \ldots.,$$

and

$$\rho\delta_1 - 2\delta_2 = \rho S_1d_0 - (\rho S_2 + 2S_2^2)\,d_1 + (\rho S_3 + 2S_3^2)\,d_2 - \ldots.;$$

---

* The construction of the analogous function

$$S_w^r \equiv \Sigma_r\alpha_1^{p_1}\alpha_2^{p_2}\ldots.\alpha_r^{p_r}, \qquad \left(\begin{matrix}\Sigma p = w \\ p_1, p_2, p_3, \ldots. p_r > 0\end{matrix}\right)$$

when expressed in terms of the literal coefficients may be exhibited as follows. Let

$$a_0^{\mu_0}a_1^{\mu_1}\ldots.a_s^{\mu_s},\ \ldots.a_t^{\mu_t}$$

be any product of coefficients of weight w; the corresponding partition of w consists of μ_1 ones, μ_2 twos and so on, and is represented by

$$(1^{\mu_1}2^{\mu_2}\ldots.s^{\mu_s}\ldots.t^{\mu_t});$$

we form the function

$$\sum_{s=1}^{s=t}\mu_s(1+x)^s = \sum_{s=1}^{s=t}\mu_s + \sum_{s=1}^{s=t}\mu_s sx + \sum_{s=1}^{s=t}\mu_s\frac{s.s-1}{2!}x^2 + \ldots. + \sum_{s=1}^{s=t}\mu_s\frac{s!}{r!\,(s-r)!}x^r + \ldots.$$
$$= W_0 + W_1x + W_2x^2 + \ldots. + W_rx^r + \ldots.,$$

and then

$$S_w^r = \Sigma(-)^{W_0+W_1+r+1}\frac{(W_0-1)!\ W_r}{\mu_1!\ \mu_2!\ldots.\mu_t!}a_0^{\mu_0}a_1^{\mu_1}\ldots.a_t^{\mu_t}.$$

As a verification, observe that

$$S_w^1 + S_w^2 + \ldots + S_w^w = \Sigma\frac{(-)^{W_0+W_1}}{\mu_1!\ \mu_2!\ldots\mu_t!}(W_0-1)!\,\{W_1 - W_2 + W_3 - \ldots + (-)^{w+1}W_w\}\,a_0^{\mu_0}a_1^{\mu_1}\ldots a_t^{\mu_t}$$
$$= \Sigma\frac{(-)^{W_0+W_1}W_0!}{\mu_1!\ \mu_2!\ldots\mu_t!}a_0^{\mu_0}a_1^{\mu_1}\ldots a_t^{\mu_t}$$
$$= h_w \text{ as should be the case.}$$

δ_1 and δ_2 are each separately reversors; expressed by means of partitions, we find

$$\delta_1 = (1)\,d_0 - (2)\,d_1 + (3)\,d_2 - (4)\,d_3 + \ldots.$$

$$\delta_2 = (1^2)\,d_1 - (21)\,d_2 + \{(31) + (2^2)\}\,d_3 - \{(41) + (32)\}\,d_4 + \ldots.$$

§4.

Operation of the Reversors upon a source which is symbolically expressed by means of Partitions.

It is well known that any source may be expressed symbolically by an aggregate of partitions of its weight, the parts of each partition being drawn from the natural numbers to w inclusive, unity alone being excluded; and further, that the highest symbolic number occurring in the aggregate represents the degree of the source in the coefficients; the operation of d_λ upon any such partition is seen by writing

$$d_\lambda \equiv \Sigma\,(-)^{W_0+W_1}\,\frac{(W_0-1)!\;W_1}{\mu_1!\,\mu_2!\ldots\mu_t!}\,D_1^{\mu_1}D_2^{\mu_2}\ldots D_t^{\mu_t}, \qquad \begin{pmatrix} W_0 = \Sigma\mu \\ W_1 = \Sigma s\mu_s \end{pmatrix}$$

where

$$D_\kappa = \frac{1}{\varkappa!}\,(a_0\partial_{a_1} + a_1\partial_{a_2} + a_2\partial_{a_3} + \ldots)^\kappa,$$

and remarking Hammond's theorem that D_κ operating upon a partition has the effect of striking out therefrom one symbolic number $\varkappa$ when such is possible, and an annihilating effect in every other case. It is hence manifest that δ_1 introduces, by the operation of its member

$$(1)\,d_0,$$

symmetric functions whose partitions contain at most one unit; also in the operation of δ_2, (1^2) will not appear as a multiplier, since in the present case $d_1 \equiv D_1 \equiv 0$, and $S_\lambda^2\,(\lambda > 2)$ is expressed by employing merely a single unit; whence δ_2 also introduces partitions which contain at most one unit, and so also does any linear function of δ_1 and δ_2.

This is obvious, too, from the consideration that any coefficient is derivable from the next succeeding one by the operation of

$$d_1,$$

the effect of which is to obliterate a non-unitary partition and to take away one part unity from all unitary partitions; we now see by a process of induction that a covariant coefficient, s removes from its source, is expressible symbolically by means of partitions, no one of which contains more than s parts equal to unity.

§5.

General Form of a Covariant.

Denote by
$$\theta_s$$
any assemblage of non-unitary positive integers, the sum of which is $w + s$, and of which no number exceeds θ in magnitude; any two covariants then assume the forms

$$(\theta_0) x_0^s + s\{(\theta_0 1) + (\theta_1)\} x_0^{s-1} x_1 + s.s-1\{(\theta_0 1^2) + (\theta_1 1) + (\theta_2)\} x_0^{s-2} x_1^2 + \ldots\ldots,$$
$$(\phi_0) x_0^\sigma + \sigma\{(\phi_0 1) + (\phi_1)\} x_0^{\sigma-1} x_1 + \sigma.\sigma-1\{(\phi_0 1^2) + (\phi_1 1) + (\phi_2)\} x_0^{\sigma-2} x_1^2 + \ldots\ldots,$$

to which, applying Ueberschiebung of order κ, there results the source

$$\begin{aligned}&(\theta_0)\{(\phi_0 1^\kappa) + (\phi_1 1^{\kappa-1}) + \ldots\ldots + (\phi_\kappa)\}\\ &-\{(\theta_0 1) + (\theta_1)\}\{(\phi_0 1^{\kappa-1}) + (\phi_1 1^{\kappa-2}) + \ldots\ldots + (\phi_{\kappa-1})\}\\ &+\ldots\ldots\\ &+(-)^\kappa\{(\theta_0 1^\kappa) + (\theta_1 1^{\kappa-1}) + \ldots\ldots + (\theta_\kappa)\}(\phi_0),\end{aligned}$$

an expression which consists of $\frac{1}{6}(\kappa + 1)(\kappa + 2)(\kappa + 3)$ binary products, and which may be put into the form:

$$\begin{aligned}&\sum_{j=0}^{j=\kappa}\{(\theta_0)(\phi_j 1^{\kappa-j}) - (\theta_0 1)(\phi_j 1^{\kappa-j-1}) + (\theta_0 1^2)(\phi_j 1^{\kappa-j-2}) - \ldots\ldots + (-)^{\kappa-j}(\theta_0 1^{\kappa-j})(\phi_j)\}\\ &-\sum_{j=0}^{j=\kappa-1}\{(\theta_1)(\phi_j 1^{\kappa-j-1}) - (\theta_1 1)(\phi_j 1^{\kappa-j-2}) + (\theta_1 1^2)(\phi_j 1^{\kappa-i-3})\\ &\qquad\qquad - \ldots\ldots + (-)^{\kappa-j-1}(\theta_1 1^{\kappa-j-1})(\phi_j)\}\\ &+\ldots\ldots\\ &+(-)^\kappa(\theta_\kappa)(\phi_0);\end{aligned}$$

or as the double sum

$$\begin{aligned}&\sum_{i=0}^{i=\kappa}\sum_{j=0}^{j=\kappa-i}(-)^i\{(\theta_i)(\phi_j 1^{\kappa-i-j}) - (\theta_i 1)(\phi_j 1^{\kappa-i-j-1})\\ &\qquad + \ldots\ldots + (-)^{\kappa-i-j+1}(\theta_i 1^{\kappa-i-j-1})(\phi_j 1) + (-)^{\kappa-i-j}(\theta_i 1^{\kappa-i-j})(\phi_j)\};\end{aligned}$$

which constitutes the partition expression of the source of the covariant which is derived from Ueberschiebung of order κ applied to the two given covariants.

The source clearly consists of $\frac{(\kappa+1)(\kappa+2)}{2}$ $(= 1 + 2 + \ldots\ldots + \kappa + 1)$ distinct parts, or members, a number which, in the general case here considered, cannot be exceeded, and which usually will not be reached.

Theorem. "Each constituent member of a source is itself a seminvariant."

For operating upon any expression of the form

$$(\theta)(\phi 1^t) - (\theta 1)(\phi 1^{t-1}) + \ldots + (-)^{t-1}(\theta 1^{t-1})(\phi 1) + (-)^t(\theta 1^t)(\phi)$$

with the annihilator $d_1 \equiv D_1$ (*ante*),

we find

$$(\theta)(\phi 1^{t-1}) - (\theta)(\phi 1^{t-1}) + (\theta 1)(\phi 1^{t-2}) - (\theta 1)(\phi 1^{t-2}) + \ldots + (-)^{t-1}(\theta 1^{t-1})(\phi) + (-)^t(\theta 1^{t-1})(\phi),$$

which vanishes, the terms destroying each other in pairs.

Each constituent member is therefore a non-unitary symmetric function, and the process of Ueberschiebung is seen to produce sources, each of which is an aggregate of symmetrical members of the same type.*

§6.

Incorporation of Two Sources.

I call, provisionally, the expression

$$(\theta_0)(\phi_0 1^\kappa) - (\theta_0 1)(\phi_0 1^{\kappa-1}) + \ldots + (-)^\kappa(\theta_0 1^\kappa)(\phi_0)$$

the incorporation of (θ_0) and (ϕ_0) of order κ, or say the κ^{th} incorporation of (θ_0) and (ϕ_0). I denote this by $|(\theta_0)(\phi_0)|^\kappa$.

From the previous section it will be seen that the κ^{th} Ueberschiebung of (θ_0) and (ϕ_0) is composed in general of

1 incorporation of order κ,
2 " " $\kappa - 1$,
3 " " $\kappa - 2$,
. . . .
$\kappa + 1$ " " 0;

---

*This theorem may be extended in the following manner. Forming the expression

$$(\theta 1^{s-1})(\phi 1^t) - s(\theta 1^s)(\phi 1^{t-1}) + \frac{s \cdot s+1}{2!}(\theta 1^{s+1})(\phi 1^{t-2}) - \ldots + (-)^t \frac{(s+t-1)!}{(s-1)!\,t!}(\theta 1^{s+t-1})(\phi),$$

and operating with D_1 we find

$$(\theta 1^{s-2})(\phi 1^t) - (s-1)(\theta 1^{s-1})(\phi 1^{t-1}) + \frac{s-1 \cdot s}{2!}(\theta 1^s)(\phi 1^{t-2}) - \ldots + (-)^t \frac{(s+t-2)!}{(s-2)!\,t!}(\theta 1^{s+t-2})(\phi),$$

which is derived from the former by writing $s-1$ for s; whence, assuming it to have been shown that the latter expression is expressible by partitions each of which contains at most $s-2$ units, it has been proved that the former is expressible by partitions each of which contains at most $s-1$ units; it has been proved to be true in the latter case when $s=2$, and hence by induction the theorem is true for any value of s.

thus a κ^{th} Ueberschiebung is composed of a κ^{th} incorporation of the same two sources, together with other incorporations of lower order; conversely, a κ^{th} incorporation is expressible by means of a κ^{th} Ueberschiebung of the same two sources together with incorporations of lower order; hence it is an immediate inference that a κ^{th} incorporation is expressible by means of a κ^{th} Ueberschiebung of the same two sources together with Ueberschiebungen of lower order. The forms which here present themselves do not in general belong to the original quantic, and for this reason, in what follows, the quantic will be supposed of indefinite order unless the contrary be expressly stated.

If in the κ^{th} incorporation of (θ_0) and (ϕ_0) we write

$$(\theta_0 1^i) = a_i,$$
$$(\phi_0 1^i) = b_i,$$

it assumes the form

$$a_0 b_\kappa - a_1 b_{\kappa-1} + a_2 b_{\kappa-2} - \dots + (-)^\kappa a_\kappa b_0,$$

which is the source of a covariant of the system of two quantics

$$(a_0,\ 1!\ a_1,\ 2!\ a_2,\ \dots)(x_0,\ x_1)^{p_1},$$
$$(b_0,\ 1!\ b_1,\ 2!\ b_2,\ \dots)(x_0,\ x_1)^{p_2},$$

from the simple fact that $\quad d_1(\theta_0 1^i) = (\theta_0 1^{i-1}),$
it is evident, *à priori*, that if

$$f(a_0,\ a_1,\ a_2,\ \dots\ b_0,\ b_1,\ b_2,\ \dots\ c_0,\ c_1,\ c_2,\ \dots)$$

be the source of any covariant of the system of quantics

$$(a_0,\ 1!\ a_1,\ 2!\ a_2,\ \dots)(x_0,\ x_1)^{p_1},$$
$$(b_0,\ 1!\ b_1,\ 2!\ b_2,\ \dots)(x_0,\ x_1)^{p_2},$$
$$(c_0,\ 1!\ c_1,\ 2!\ c_2,\ \dots)(x_0,\ x_1)^{p_3},$$

. . . .

. . . .

. . . .

the expression

$$f\{(\theta_0),\ (\theta_0 1),\ (\theta_0 1^2)\ \dots\ (\phi_0),\ (\phi_0 1),\ (\phi_0 1^2)\ \dots\ (\psi_0),\ (\psi_0 1),\ (\psi_0 1^2)\ \dots\}$$

must be a seminvariant; the partitions refer as usual to the same quantic of infinite order.

The process of incorporation is thus a very particular case of this generalization.

§7.

Binomial Syzygies.

If θ_0, ϕ_0, ψ_0 retain their original significations, we may write down the absolute identity

$$\begin{vmatrix} (\theta_0), & (\phi_0), & (\psi_0) \\ (\theta_0), & (\phi_0), & (\psi_0) \\ (\theta_0 1), & (\phi_0 1), & (\psi_0 1) \end{vmatrix} = 0,$$

that is,

$$\begin{aligned} &(\theta_0)\{(\phi_0)(\psi_0 1) - (\phi_0 1)(\psi_0)\} \\ +&(\phi_0)\{(\psi_0)(\theta_0 1) - (\psi_0 1)(\theta_0)\} \\ +&(\psi_0)\{(\theta_0)(\phi_0 1) - (\theta_0 1)(\phi_0)\} = 0; \end{aligned}$$

this identity manifestly represents a syzygy between the three compound seminvariants

$$\begin{aligned} &(\theta_0)\{(\phi_0)(\psi_0 1) - (\phi_0 1)(\psi_0)\}, \\ &(\phi_0)\{(\psi_0)(\theta_0 1) - (\psi_0 1)(\theta_0)\}, \\ &(\psi_0)\{(\theta_0)(\phi_0 1) - (\theta_0 1)(\phi_0)\}; \end{aligned}$$

an application to the theory of perpetuants may be at once shown, for write

$$(\theta_0, \phi_0, \psi_0) = (2^\kappa, 2^\lambda, 2^\mu),$$

and then

$$\begin{aligned} &(2^\kappa)\{(2^\lambda)(2^\mu 1) - (2^\lambda 1)(2^\mu)\} \\ +&(2^\lambda)\{(2^\mu)(2^\kappa 1) - (2^\mu 1)(2^\kappa)\} \\ +&(2^\mu)\{(2^\kappa)(2^\lambda 1) - (2^\kappa 1)(2^\lambda)\} = 0; \end{aligned}$$

the expression

$$(2^\lambda)(2^\mu 1) - (2^\lambda 1)(2^\mu),$$

as well as the other two expressions obtained from it by the cyclical substitution $(\varkappa\lambda\mu)$, consists, when developed in a series of monomial symmetric functions, of quartic and cubic perpetuants; as a consequence, the identity represents a sextic syzygy between binary products of quartic and quadric perpetuants.

Putting now $\mu = 0$, there results the identity

$$\begin{aligned} &(2^\kappa)\{(2^\lambda)(1) - (2^\lambda 1)\} \\ +&(2^\lambda)\{(2^\kappa 1) - (1)(2^\kappa)\} \\ +&\quad\{(2^\kappa)(2^\lambda 1) - (2^\kappa 1)(2^\lambda)\} = 0; \end{aligned}$$

herein

$$\begin{aligned} (2^\lambda)(1) - (2^\lambda 1) &= (3 2^{\lambda-1}), \\ (2^\kappa 1) - (1)(2^\kappa) &= -(3 2^{\kappa-1}); \end{aligned}$$

whence

$$(3 2^{\lambda-1})(2^\kappa) - (3 2^{\kappa-1})(2^\lambda) = (2^\lambda)(2^\kappa 1) - (2^\lambda 1)(2^\kappa);$$

remarking that the right-hand side of this identity consists wholly of quartic and cubic perpetuants, we see that we have here exhibited a quintic syzygy between the binary products on the left-hand side.

It is easy to see otherwise that we have the congruence

$$(32^{\lambda-1})(2^{\kappa})-(32^{\kappa-1})(2^{\lambda})\equiv 0 \bmod a,$$

for

$$(32^{\lambda-1})(2^{\kappa})-(32^{\kappa-1})(2^{\lambda})$$

certainly contains when developed no symbolic numbers greater than 5; but also

$$D_5\{(32^{\lambda-1})(2^{\kappa})-(32^{\kappa-1})(2^{\lambda})\}=(2^{\lambda-1})(2^{\kappa-1})-(2^{\kappa-1})(2^{\lambda-1})=0;$$

so that the symbolic number 5 does not occur; and also

$$D_4\{(32^{\lambda-1})(2^{\kappa})-(32^{\kappa-1})(2^{\lambda})\}=(32^{\lambda-2})(2^{\kappa-1})-(32^{\kappa-2})(2^{\lambda-1}),$$

which does not vanish; the symbolic number 4 does occur therefore in the development, and this establishes that the expression

$$(32^{\lambda-1})(2^{\kappa})-(32^{\kappa-1})(2^{\lambda})$$

contains as a factor the seminvariant a raised to the first and to no higher power. The above written congruence is thus verified.

Moreover, the syzygy that has been obtained, includes, for suitable values of κ and λ, every quintic syzygy in the theory of perpetuants.

To establish this it is merely necessary to note the effect of operating upon the left-hand side with D_5; it is then seen that to a syzygy, of weight w, corresponds invariably either a quadric or a quartic compound form which is not a perfect square; since the generating function for these forms is

$$\frac{x^2}{1-x^2}+\frac{x^6}{1-x^2.1-x^4}=\frac{x^2}{1-x^2.1-x^4},$$

it at once follows that the generating function for independent syzygies is

$$x^5.\frac{x^2}{1-x^2.1-x^4}=\frac{x^7}{1-x^2.1-x^4},$$

which we, otherwise, know to represent the total number (cf. Hammond, *Amer. Journ. Math.*, Vol. 4, p. 218).

Reverting to the above general sextic syzygy, let us suppose

$$\kappa \geqq \lambda > \mu,$$

and then operate upon the identity with

$$D_6^{\mu};$$

we thus obtain

$$\begin{aligned}&(2^{\kappa-\mu})\{(2^{\lambda-\mu})(1)-(2^{\lambda-\mu}1)\}\\+&(2^{\lambda-\mu})\{(2^{\kappa-\mu}1)-(1)(2^{\kappa-\mu})\}\\+&\{(2^{\kappa-\mu})(2^{\lambda-\mu}1)-(2^{\kappa-\mu}1)(2^{\lambda-\mu})\}=0,\end{aligned}$$

which is equivalent to the result obtained by simply giving μ a zero value in the original identity. In Professor Cayley's language, the effect of the operation of D_6 is, in the present instance, to decapitate the identity. We have in fact here performed μ successive operations of decapitation, and we have thereby arrived at the most general quintic syzygy; it hence follows that the sextic syzygies we are now discussing are those derivable by an infinite number of processes of (4.2) capitation of the quintic syzygies. Their generating is manifestly

$$\frac{x^7}{(1-x^2)(1-x^4)} \cdot \frac{x^6}{1-x^6} = \frac{x^{13}}{(1-x^2)(1-x^4)(1-x^6)};$$

altogether, syzygies, enumerated by means of the generating function

$$\frac{x^7}{(1-x^2)(1-x^4)(1-x^6)}$$

are exhibited in a crystalline form by means of a single absolute identity.

As another example take

$$(\theta_0, \phi_0, \psi_0) = (2^\kappa, a_0, 3^\lambda 2^\mu),$$

and then

$$\begin{aligned}&(2^\kappa)\{(3^\lambda 2^\mu 1) - (1)(3^\kappa 2^\lambda)\}\\ &+ (3^\lambda 2^\mu)\{(2^\kappa)(1) - (2^\kappa 1)\}\\ &+ \{(3^\lambda 2^\mu)(2^\kappa 1) - (3^\lambda 2^\mu 1)(2^\kappa)\} = 0,\end{aligned}$$

an identity which will be shown to represent every binomial sextic syzygy in the theory of perpetuants. It may in fact be written

$$(43^{\lambda-1}2^\mu)(2^\kappa) - (3^\lambda 2^\mu)(3\,2^{\kappa-1}) = \{(3^\lambda 2^\mu)(2^\kappa 1) - (3^\lambda 2^\mu 1)(2^\kappa)\} - (\lambda+1)(3^{\lambda+1}2^{\mu-1})(2^\kappa),$$

and hence a reference to my first paper on perpetuants in the *American Journal of Mathematics* (Vol. 7, p. 26) will show that the statement is true.

The fact is, that all simple syzygies may be exhibited and enumerated in a precisely similar manner, but it is not appropriate to further multiply examples of the method. The discussion of the capitation syzygies in general is, from their inherent nature, a far more difficult matter, and all that will be done here subsequently will be to indicate to what extent progress has been made.

§8.

Particular Forms of Incorporation.

Consider the form

$$|(2^j),\, a_0|^m \equiv (2^j)(1^m) - (2^j 1)(1^{m-1}) + (2^j 1^2)(1^{m-2}) - \ldots\ldots + (-)^m (2^j 1^m),$$

and the result of expanding it in a series of monomials. Since

$$D_3\,|(2^j),\,a_0|^m = |(2^{j-1}),\,a_0|^{m-1},$$

it appears that having expanded $|(2^{j-1}),\,a_0|^{m-1}$, we have merely to capitate each partition therein presenting itself with the symbolic number 3, in order at once to obtain all those partitions in the expansion of

$$|(2^j),\,a_0|^m$$

in which a symbolic number 3 appears; thus, since

$$|(2^{j-1}),\,a_0|^1 = (3\,2^{j-2}),$$

we at once find

$$|(2^j),\,a_0|^2 = (3^2 2^{j-2}) + \ldots\ldots;$$

hence, to obtain the general formula, it is only necessary to find that term in the expansion which is made up wholly of twos. Such a term only exists for an even value of m, and the only product of partitions in

$$|(2^j),\,a_0|^m$$

which can possibly give rise to it is

$$(-)^{\frac{1}{2}m}(2^j 1^{\frac{1}{2}m})(1^{\frac{1}{2}m}),$$

the expansion of which includes the term

$$(-)^{\frac{1}{2}m}\frac{\left(j+\frac{1}{2}m\right)!}{\frac{m}{2}!\,j!}\,(2^{j+\frac{1}{2}m})$$

(see Cayley, *A. J. of M.*, Vol. 7, p. 1); hence, by an easy process, we reach the formula

$$|(2^j),\,a_0|^m = \sum_{s=0}^{s \overline{\overline{<}} \frac{1}{2}m} (-)^s\,\frac{(j-m+3s)!}{s!\,(j-m+2s)!}\,(3^{m-2s}2^{j-m+3s}),$$

which exhibits the m^{th} incorporation of the general quadric form (2^j) with the simplest seminvariant a_0. Conversely, we can express any single partition cubic form in terms of incorporation, s, thus

$$(3\,2^\kappa) = |(2^{\kappa+1}),\,a_0|^1,$$
$$(3^2 2^\kappa) = |(2^{\kappa+2}),\,a_0|^2 + (\kappa+3)\,|(2^{\kappa+3}),\,a_0|^0,$$

and in general

$$(3^p 2^\kappa) = \sum_{s=0}^{s \overline{\overline{<}} \frac{1}{2}p} \frac{(\kappa+3s).(\kappa+s-1)!}{\kappa!\,s!}\,|(2^{\kappa+p+s}),\,a_0|^{p-2s}.$$

Consider now the incorporation $|(3^\kappa 2^\lambda),\ a_0|^m$. This form is a quartic seminvariant; for reasons similar to those above given in the previous case, and which are consequent upon the relation

$$D_4|(3^\kappa 2^\lambda),\ a_0|^m = |(3^{\kappa-1}2^\lambda),\ a_0|^{m-1}.$$

We need only calculate those terms in the expansion which do not contain a symbolic number 4. The product $(-)^s(3^\kappa 2^\lambda 1^s)(1^{m-s})$ which occurs in $|(3^\kappa 2^\lambda),\ a_0|^m$, when multiplied out, only produces one such partition, and that one is

$$(-)^s(3^{\kappa+m-2s}2^{\lambda-m+3s});$$

the numerical coefficient attached to it is (vide Cayley loc. cit.)

$$\frac{(\kappa+m-2s)!\ (\lambda-m+3s)!}{\kappa!\ s!\ (m-2s)!\ (\lambda-m+2s)!},$$

hence the following formula:

$$|(3^\kappa 2^\lambda),\ a_0|^m =$$

$$\sum_{p=0}^{p=m}\ \sum_{s=0}^{s \overline{\gtrless} \frac{1}{2}(m-p)} \frac{(-)^s(\kappa+m-2p-2s)!\ (\lambda-m+p+3s)!}{(\kappa-p)!\ s!\ (m-p-2s)!\ (\lambda-m+p+2s)!}(4^p 3^{\kappa+m-2p-2s}2^{\lambda-m+p+3s}).$$

The incorporation of two quadric forms may be expressed similarly. If we wish to incorporate the seminvariant

$$|(2^\kappa),\ a_0|^m \equiv (2^\kappa)(1^m) - (2^\kappa 1)(1^{m-1}) + \ldots\ldots + (-)^m(2^\kappa 1^m)$$

with a_0, we may proceed in more than one way, but only one really distinct result is obtainable for each order of incorporation. Remembering that the expression

$$(2^\kappa 1^s)(1^m) - (s+1)(2^\kappa 1^{s+1})(1^{m-1}) + \frac{(s+1)(s+2)}{2!}(2^\kappa 1^{s+2})(1^{m-2})$$
$$- \ldots\ldots + (-)^m \frac{(s+m)!}{s!\ m!}(2^\kappa 1^{s+m})$$

consists, when expanded, of partitions containing at most s units, we may form the incorporation of order t:

$$\{(2^\kappa)(1^m) - (2^\kappa 1)(1^{m-1}) + \ldots\ldots + (-)^m(2^\kappa 1^m)\}(1^t)$$
$$- \{(2^\kappa 1)(1^m) - 2(2^\kappa 1^2)(1^{m-1}) + \ldots\ldots + (-)^m m(2^\kappa 1^{m+1})\}(1^{t-1})$$
$$+ \{(2^\kappa 1^2)(1^m) - 3(2^\kappa 1^3)(1^{m-1}) + \ldots\ldots + (-)^m \frac{m.m+1}{2!}(2^\kappa 1^{m+2})\}(1^{t-2})$$
$$- \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+ (-)^t\{(2^\kappa 1^t)(1^m) - (t+1)(2^\kappa 1^{t+1})(1^{m-1}) + \ldots\ldots + (-)^m \frac{(m+t-1)!}{t!\ (m-1)!}(2^\kappa 1^{m+t})\},$$

which is

$$(2^\kappa)(1^m)(1^t) - (2^\kappa 1)\, D_1\{(1^m)(1^t)\} + (2^\kappa 1^2)\, D_1^2\{(1^m)(1^t)\} \\ - \ldots . + (-)^{m+t}(2^\kappa 1^{m+t})\, D_1^{m+t}\{(1^m)(1^t)\},$$

as may be seen by making a rearrangement of terms. Now

$$(1^m)(1^t) = (2^t 1^{m-t}) + (m-t+2)(2^{t-1}1^{m-t+2}) \\ + \frac{(m-t+3)(m-t+4)}{2!}(2^{t-2}1^{m-t+4}) + \ldots .$$

$$\therefore D_1^s\{(1^m)(1^t)\} = (2^t 1^{m-t-s}) + (m-t+2)(2^{t-1}1^{m-t-s+2}) \\ + \frac{(m-t+3)(m-t+4)}{2!}(2^{t-2}1^{m-t-s+4}) + \ldots .,$$

whence the incorporation of $|(2^\kappa),\, a_0|^m$ with a_0 may be written:

$$(2^\kappa)(2^t 1^{m-t}) - (2^\kappa 1)(2^t 1^{m-t-1}) + (2^\kappa 1^2)(2^t 1^{m-t-2}) - \ldots . + (-)^{m-t}(2^\kappa 1^{m-t})(2^t) \\ + (m-t+2)\{(2^\kappa)(2^{t-1}1^{m-t+2}) - (2^\kappa 1)(2^{t-1}1^{m-t+1}) + \ldots . + (-)^{m-t}(2^\kappa 1^{m-t+2})(2^{t-1})\} \\ + \frac{(m-t+4)!}{2!\,(m-t+2)!}\{(2^\kappa)(2^{t-2}1^{m-t+4}) - \ldots .\} \\ + \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

or finally as

$$|(2^\kappa),\, (2^t)|^{m-t} + (m-t+2)|(2^\kappa),\, (2^{t-1})|^{m-t+2} \\ + \frac{(m-t+4)!}{2!\,(m-t+2)!}|(2^\kappa),\, (2^{t-2})|^{m-t+4} + \ldots . + \frac{(m+t)!}{t!\,m!}|(2^\kappa),\, a_0|^{m+t};$$

thus the incorporation which is competent to produce every quartic form is expressible by means of incorporations of quadric forms. Generally it will be found that every form of degree n can be obtained by incorporations of quadric forms with forms of degree $n-2$.

§9.

Capitation Syzygies.

Consider the two cubic forms

$$|(2^\kappa),\, a_0|^1 \equiv (2^\kappa)(1) - (2^\kappa 1),$$
$$|(2^\mu),\, a_0|^1 \equiv (2^\mu)(1) - (2^\mu 1),$$

from which

$$|(2^\kappa),\, a_0|^1 . |(2^\mu),\, a_0|^1 = (1)^2(2^\kappa)(2^\mu) - (1)\{(2^\kappa)(2^\mu 1) + (2^\kappa 1)(2^\mu)\} + (2^\kappa 1)(2^\mu 1), \\ = (2)(2^\kappa)(2^\mu) + 2(1^2)(2^\kappa)(2^\mu) \\ - (1)\{(2^\kappa)(2^\mu 1) + (2^\kappa 1)(2^\mu)\} + (2^\kappa 1)(2^\mu 1);$$

hence

$$|(2^\kappa), a_0|^1 . |(2^\mu), a_0|^1 - (2)(2^\kappa)(2^\mu) = (2^\kappa)\,|(2^\mu), a_0|^2 + (2^\mu)\,|(2^\kappa), a_0|^2 - |(2^\kappa), (2^\mu)|^2;$$

wherein the sinister contains sextic compounds and the dexter consists of quintic compounds, quartic compounds, and quartic, cubic, and quadric perpetuants.

We have here, therefore, a syzygy. As a simple example, put $\kappa = \mu = 1$, and then

$$(3)(3) - (2)(2)(2) = (2)\{-2(2^2)\} + (2)\{-2(2^2)\} + (42) + 2(3^2) + 6(2^3),$$

or $$(3)^2 - (2)^3 = -3(42) + 2(3^2) - 6(2^3),$$

and this is the syzygy which yields the discriminant of the cubic. This general result arose by putting

$$(1)^2 = (2) + 2(1^2),$$

or we may say it came from the congruence

$$(1)^2 - (2) \equiv 0 \bmod a_0.$$

Consider next the congruence

$$(1^2)^2 - 2(1)(1^3) - (2^2) \equiv 0 \bmod a_0;$$

proceed as follows:

$$\begin{aligned}
&|(2^\kappa), a_0|^4 . |(2^\mu), a_0|^0 \\
&= (1^4)(2^\kappa)(2^\mu) - (1^3)(2^\kappa 1)(2^\mu) + (1^2)(2^\kappa 1^2)(2^\mu) - (1)(2^\kappa 1^3)(2^\mu) + (2^\kappa 1^4)(2^\mu),
\end{aligned}$$

$$\begin{aligned}
&|(2^\kappa), a_0|^3 . |(2^\mu), a_0|^1 \\
&= (1)(1^3)(2^\kappa)(2^\mu) - (1)(1^2)(2^\kappa 1)(2^\mu) + (1^2)(2^\kappa 1)(2^\mu 1) - (1)(2^\kappa 1^3)(2^\mu) + (2^\kappa 1^3)(2^\mu 1) \\
&\qquad - (1^3)(2^\kappa)(2^\mu 1) + (1)^2(2^\kappa 1^2)(2^\mu) - (1)(2^\kappa 1^2)(2^\mu 1),
\end{aligned}$$

$$\begin{aligned}
&|(2^\kappa), a_0|^2 . |(2^\mu), a_0|^2 \\
&= (1^2)^2(2^\kappa)(2^\mu) - (1)(1^2)(2^\kappa)(2^\mu 1) + (1^2)(2^\kappa)(2^\mu 1^2) - (1)(2^\kappa 1)(2^\mu 1^2) + (2^\kappa 1^2)(2^\mu 1^2) \\
&\qquad - (1)(1^2)(2^\kappa 1)(2^\mu) + (1^2)(2^\kappa 1^2)(2^\mu) - (1)(2^\kappa 1^2)(2^\mu 1) \\
&\qquad\qquad + (1)^2(2^\kappa 1)(2^\mu 1),
\end{aligned}$$

$$\begin{aligned}
&|(2^\kappa), a_0|^1 . |(2^\mu), a_0|^3 \\
&= (1)(1^3)(2^\kappa)(2^\mu) - (1)(1^2)(2^\kappa)(2^\mu 1) + (1^2)(2^\kappa 1)(2^\mu 1) - (1)(2^\kappa)(2^\mu 1^3) + (2^\kappa 1)(2^\mu 1^3) \\
&\qquad - (1^3)(2^\kappa 1)(2^\mu) + (1)^2(2^\kappa)(2^\mu 1^2) - (1)(2^\kappa 1)(2^\mu 1^2),
\end{aligned}$$

$$\begin{aligned}
&|(2^\kappa), a_0|^0 . |(2^\mu), a_0|^4 \\
&= (1^4)(2^\kappa)(2^\mu) - (1^3)(2^\kappa)(2^\mu 1) + (1^2)(2^\kappa)(2^\mu 1^2) - (1)(2^\kappa)(2^\mu 1^3) + (2^\kappa)(2^\mu 1^4).
\end{aligned}$$

Consider these five identities and subtract the second and fourth from the sum of the first, third and fifth; in the result it will be seen that the second and fourth columns of terms on the right-hand side vanish identically; the third column of terms easily reduces to

$$-(2)\,|(2^\kappa),\,(2^\mu)|^2,$$

and the fifth column to

$$|(2^\kappa),\,(2^\mu)|^4;$$

hence

$$\sum_{\alpha=0}^{\alpha=4}(-)^\alpha\,|(2^\kappa),\,a_0|^\alpha.\,|(2^\mu),\,a_0|^{4-\alpha}$$

$$=(2^2)(2^\kappa)(2^\mu)-(2)\,|(2^\kappa),\,(2^\mu)|^2+|(2^\kappa),\,(2^\mu)|^4,$$

or finally, this is a formula representing syzygies between (4.2), (3.3) and (2.2.2) sextic compounds of perpetuants.

It is manifest that this process, here given at length, is perfectly general; the result is that the congruence ($p>1$)

$$(1^p)^2-2(1^{p-1})(1^{p+1})+2\,(1^{p-2})(1^{p+2})-\ldots.+(-)^{p+1}2(1)(1^{2p-1})-(2^p)\equiv 0 \bmod a_0$$

leads to the formula of syzygies

$$\sum_{\alpha=0}^{\alpha=2p}(-)^{\alpha+p}\,|(2^\kappa),\,a_0|^\alpha.\,|(2^\mu),\,a_0|^{2p-\alpha}$$

$$=(2^p)\,|(2^\kappa),\,(2^\mu)|^0-(2^{p-1})\,|(2^\kappa),\,(2^\mu)|^2+(2^{p-2})\,|(2^\kappa),\,(2^\mu)|^4-\ldots.$$
$$+(-)^p\,|(2^\kappa),\,(2^\mu)|^{2p},$$

or, as it may be written,

$$\sum_{\alpha=0}^{\alpha=2p}(-)^{\alpha+p}\,|(2^\kappa),\,a_0|^\alpha.\,|(2^\mu),\,a_0|^{2p-\alpha}=\sum_{\beta=0}^{\beta=p}(-)^\beta\,(2^{p-\beta})\,|(2^\kappa),\,(2^\mu)|^{2\beta},$$

denoting a number of complete syzygies of degree 6 and of weight $2(\kappa+\mu+p)$.

§10.

Consider next the formation of syzygies of uneven weight. Since

$$|(2^\kappa),\,a_0|^2=(1^2)(2^\kappa)-(1)(2^\kappa 1)+(2^\kappa 1^2),$$
$$|(2^\mu),\,a_0|^1=(1)(2^\mu)-(2^\mu 1),$$
$$\therefore\ |(2^\kappa),\,a_0|^2.\,|(2^\mu),\,a_0|^1-|(2^\kappa),\,a_0|^1.\,|(2^\mu),\,a_0|^2-(1)^2\{(2^\kappa)(2^\mu 1)$$
$$-(2^\kappa 1)(2^\mu)\}\equiv 0 \bmod a_0,$$

or $$|(2^\kappa),\,a_0|^2.\,|(2^\mu),\,a_0|^1-|(2^\kappa),\,a_0|^1.\,|(2^\mu),\,a_0|^2-(2)\,|(2^\kappa),\,(2^\mu)|^1\equiv 0 \bmod a_0.$$

As a verification, observe that the operation of D_6 on the sinister gives

$$|(2^{\kappa-1}),\, a_0|^1\,(2^{\mu-1}) - (2^{\kappa-1})\,|(2^{\mu-1}),\, a_0|^1 - |(2^{\kappa-1}),\, (2^{\mu-1})|^1,$$

which obviously vanishes identically.

Proceed now as follows:

$$\begin{aligned}
&|(2^{\kappa}),\, a_0|^4.\,|(2^{\mu}),\, a_0|^1\\
&\quad = (1)(1^4)(2^{\kappa})(2^{\mu}) \quad - (1)(1^3)(2^{\kappa}1)(2^{\mu}) + (1)(1^2)(2^{\kappa}1^2)(2^{\mu}) - (1)^2(2^{\kappa}1^3)(2^{\mu}) + \ldots.,\\
&|(2^{\kappa}),\, a_0|^3.\,|(2^{\mu}),\, a_0|^2\\
&\quad = (1^2)(1^3)(2^{\kappa})(2^{\mu}) - (1^2)^2(2^{\kappa}1)(2^{\mu}) \quad + (1)(1^2)(2^{\kappa}1^2)(2^{\mu}) - (1)^2(2^{\kappa}1^2)(2^{\mu}1) + \ldots.\\
&\qquad\qquad\qquad - (1)(1^3)(2^{\kappa})(2^{\mu}1) + (1)(1^2)(2^{\kappa}1)(2^{\mu}1),\\
&|(2^{\kappa}),\, a_0|^2.\,|(2^{\mu}),\, a_0|^3\\
&\quad = (1^2)(1^3)(2^{\kappa})(2^{\mu}) - (1^2)^2(2^{\kappa})(2^{\mu}1) \quad + (1)(1^2)(2^{\kappa})(2^{\mu}1^2) - (1)^2(2^{\kappa}1)(2^{\mu}1^2) + \ldots.\\
&\qquad\qquad\qquad - (1)(1^3)(2^{\kappa}1)(2^{\mu}) + (1)(1^2)(2^{\kappa}1)(2^{\mu}1),\\
&|(2^{\kappa}),\, a_0|^1.\,|(2^{\mu}),\, a_0|^4\\
&\quad = (1)(1^4)(2^{\kappa})(2^{\mu}) - (1)(1^3)(2^{\kappa})(2^{\mu}1) \quad + (1)(1^2)(2^{\kappa})(2^{\mu}1^2) - (1)^2(2^{\kappa})(2^{\mu}1^3) + \ldots.
\end{aligned}$$

where on the right-hand side the portions omitted contain a_0 as a factor. Subtracting the second and fourth of these equations from the sum of the first and third, it will be seen that the first and third of the columns of terms on the right-hand side vanish identically, and we find

$$\begin{aligned}
&\sum_{\alpha=1}^{\alpha=4}(-)^{\alpha}\,|(2^{\kappa}),\, a_0|^{\alpha}.\,|(2^{\mu}),\, a_0|^{5-\alpha}\\
&\qquad = -\{(1^2)^2 - 2\,(1)(1^3)\}\{(2^{\kappa})(2^{\mu}1) - (2^{\kappa}1)(2^{\mu})\} + (1)^2\,|(2^{\kappa}),\, (2^{\mu})|^3 + \ldots.,
\end{aligned}$$

whence

$$\sum_{\alpha=1}^{\alpha=4}(-)^{\alpha}\,|(2^{\kappa}),\, a_0|^{\alpha}.\,|(2^{\mu}),\, a_0|^{5-\alpha} + (2^2)\,|(2^{\kappa}),\, (2^{\mu})|^1 - (2)\,|(2^{\kappa}),\, (2^{\mu})|^3 \equiv 0 \bmod a_0,$$

representing another batch of sextic syzygies.

The process employed is perfectly general, and enables us to write down the syzygy of odd weight:

$$\sum_{\alpha=1}^{\alpha=2p}(-)^{\alpha}\,|(2^{\kappa}),\, a_0|^{\alpha}.\,|(2^{\mu}),\, a_0|^{2p+1-\alpha} + \sum_{\beta=1}^{\beta=p}(-)^{\beta}(2^{\beta})\,|(2^{\kappa}),\, (2^{\mu})|^{2p+1-2\beta} \equiv 0 \bmod a_0;$$

from this syzygy, in the form of a congruence, we can at once obtain the complete syzygy; for, operating upon the left-hand side of the congruence with D_6, we must obtain an equation, thus

$$\sum_{\alpha=1}^{\alpha=2p}(-)^{\alpha}\,|\,(2^{\kappa-1}),\ a_0\,|^{\alpha-1}.\,|\,(2^{\mu-1}),\ a_0\,|^{2p-\alpha}$$

$$+\sum_{\beta=1}^{\beta=p}(-)^{\beta}(2^{\beta-1})\,|\,(2^{\kappa-1}),\ (2^{\mu-1})\,|^{2p+1-2\beta}=0,$$

and herein putting

$$(\kappa,\mu,p,\alpha,\beta)=(\kappa+1,\mu+1,p+1,\alpha+1,\beta+1),$$

we obtain

$$\sum_{\alpha=0}^{\alpha=2p+1}(-)^{\alpha}\,|\,(2^{\kappa}),\ a_0\,|^{\alpha}.\,|\,(2^{\mu}),\ a_0\,|^{2p+1-\alpha}+\sum_{\beta=0}^{\beta=p}(-)^{\beta}(2^{\beta})\,|\,(2^{\kappa}),\ (2^{\mu})\,|^{2p+1-2\beta}=0,$$

which is the complete syzygy of degree 6 and weight $2(\kappa+\mu+p)+1$.

Finally, it may be remarked that, in this paper, the complete expression has been exhibited of a number of sextic syzygies which are enumerated by the generating function $\dfrac{x^6+x^9+x^{11}+x^{12}+x^{13}+x^{14}+x^{15}+x^{17}}{(1-x^2)(1-x^4)(1-x^6)}$; syzygies $\dfrac{x^{18}+x^{20}+x^{23}}{1-x^2.1-x^4.1-x^6}$ remain to be exhibited, but at present I do not see how to effect this.

ROYAL MILITARY ACADEMY, WOOLWICH, July 22d, 1887.

Démonstration directe de la formule Jacobienne de la transformation cubique.

NOTE DE L'ABBÉ FAÀ DE BRUNO.

D'après les *Fundamenta*, en supposant qu'il existe l'équation differentielle

$$\frac{dy}{\sqrt{(1-y^2)(1-\lambda^2y^2)}}=\frac{1}{\mu}\,\frac{dx}{\sqrt{(1-x^2)(1-\varkappa^2x^2)}}, \tag{1}$$

en posant $y=S\left(\frac{u}{\mu},\ \lambda\right)$, $x=S(u)$,[*] on aura pour la première transformation réelle

$$S\left(\frac{u}{\mu},\ \lambda\right)=\frac{S(u)}{\mu}\prod_1^{n-1}{}_s\,\frac{1-\dfrac{S^2(u)}{S^2\left(\dfrac{2sK}{n}\right)}}{1-\varkappa^2S^2\left(\dfrac{2sK}{n}\right)}. \tag{2}$$

Lorsque $n=3$, il vient

$$S\left(\frac{u}{\mu},\ \lambda\right)=\frac{S(u)}{\mu}\,\frac{1-\dfrac{S^2(u)}{S^2\left(\dfrac{2K}{3}\right)}}{1-\varkappa^2S^2(u)\,S^2\left(\dfrac{2K}{3}\right)}. \tag{3}$$

C'est cette formule qu'il s'agit de trouver directement.

À cet effet je suppose que d'après Jacobi on ait sous la main les équations pour la transformation cubique

$$y=x\,\frac{a_0+a_1x^2}{1+b_1x^2}, \tag{4}$$

$$\left\{\begin{aligned} &a_0=1+2\alpha,\ \ b_1=\alpha(\alpha+2),\ \ a_1=\alpha^2,\ \ \alpha^2=\sqrt{\frac{\varkappa^3}{\lambda}},\\ &\qquad\qquad\sqrt[4]{\varkappa}=u,\ \ \sqrt[4]{\lambda}=v,\ \ \alpha=\frac{v^3}{u}, \end{aligned}\right. \tag{5}$$

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[*] La notation S signifie *sinam*, ou *sn* employée par d'autres. C'est celle que j'adopte dans mon *Traité complet sur les Fonctions elliptiques*, sous presse ; notation qu'on trouvera très utile par sa concision et sa clarté..

ainsi que l'équation modulaire

$$u^4 - v^4 + 2uv(1 - u^2v^2) = 0. \tag{6}$$

Observons d'abord que l'équation (4) peut s'écrire

$$y = a_0 x \frac{1 + \frac{a_1}{a_0} x^2}{1 + b_1 x^2},$$

et qu'on a d'après les relations connues de Jacobi,

$$b_1 = \frac{a_0}{a_1} \varkappa^2.$$

Par consequent il vient

$$y = a_0 x \frac{1 + \frac{a_1}{a_0} x^2}{1 + \frac{a_0}{a_1} \varkappa^2 x^2},$$

ou encore, en remarquant que $\mu = \frac{1}{a_0}$, et en posant $\beta^2 = -\frac{a_0}{a_1}$,

$$y = \frac{1}{\mu} \frac{x\left(1 - \frac{x^2}{\beta^2}\right)}{1 - \varkappa^2\beta^2x^2}. \tag{7}$$

En comparant avec la (3) il faudra que $\beta = S\left(\frac{2K}{3}\right)$, et alors l'équation (3) est démontrée. À cet effet remarquons qu'on a d'après (5),

$$\beta^2 = -\frac{a_0}{a_1} = -\frac{1 + 2\alpha}{\alpha^2} = -\frac{v + 2u^3}{u^6} v.$$

Posons $\beta^2 = z$, nous aurons l'équation

$$v^2 + 2vu^3 + u^6 z = 0, \tag{8}$$

tandis que l'équation modulaire correspondante (6) est

$$v^4 + 2u^3v^3 - 2uv - u^4 = 0,$$

équation qui à l'aide de la (8) devient

$$u^6 z v^2 + 2uv + u^4 = 0. \tag{9}$$

En éliminant v entre des deux équations (8) et (9) il vient

$$4(u^8 z - 1)(z - 1) - (1 - u^8 z^2)^2 = 0,$$

ou simplement, en remplaçant z par β^2, u^8 par $\varkappa$,

$$3 - 4\beta^2 (1 + \varkappa^2) + 6k^2\beta^4 - \varkappa^4\beta^8 - 0, \tag{10}$$

qui est précisément le numérateur de l'expression de $S(3\alpha)$, en posant $\beta = S(\alpha)$.

Or si $3\alpha = 2K$, $S(3\alpha)$ s'annulle ; et par conséquent, $\alpha = \frac{2K}{3}$ est une racine de l'équation (10); donc $\beta = S\left(\frac{2K}{3}\right)$.

Je profite de cette occasion pour faire voir comment on peut arriver simplement à un resultat de Mr. Cayley consigné dernièrement dans le v. 9 de ce Journal relativement à l'équation modulaire de 3^e ordre,

$$u^{16} + v^{16} + 12(v^{12}u^4 + u^{12}v^4) - 16(v^4u^4 + v^{12}u^{12}) + 6u^8v^8. \tag{11}$$

En posant

$$2\alpha = \varkappa + \frac{1}{\varkappa},\ 2\beta = \lambda + \frac{1}{\lambda},\ \nu = \frac{1}{\mu}\sqrt{\frac{\lambda}{\varkappa}},$$

et en écrivant l'équation differentielle (1) sous la forme

$$\frac{dy}{\sqrt{1 - 2\beta y + y^4}} = \frac{\nu dx}{\sqrt{1 - 2\alpha x^2 + x^4}},$$

il s'agit de savoir ce que devient l'équation (11) en fonction de α et β.

A cet effet pour plus de clarté faisons pour le moment $\varkappa = u^4 = \mathrm{u}$, $\lambda = v^4 = \mathrm{v}$, l'équation (11) deviendra

$$\mathrm{u}^4 + \mathrm{v}^4 + 12(\mathrm{uv}^3 + \mathrm{vu}^3) - 16(\mathrm{u}^3\mathrm{v}^3 + \mathrm{uv}) + 6\mathrm{u}^2\mathrm{v}^2 = 0. \tag{12}$$

Si on la divise par $\mathrm{u}^2\mathrm{v}^2$ on aura

$$\frac{\mathrm{u}^2}{\mathrm{v}^2} + \frac{\mathrm{v}^2}{\mathrm{u}^2} + 12\left(\frac{\mathrm{v}}{\mathrm{u}} + \frac{\mathrm{u}}{\mathrm{v}}\right) - 16\left(\mathrm{uv} + \frac{1}{\mathrm{uv}}\right) + 12 = 0. \tag{13}$$

Mais on a

$$\mathrm{u} + \frac{1}{\mathrm{u}} = 2\alpha, \qquad \mathrm{v} + \frac{1}{\mathrm{v}} = 2\beta,$$

$$\mathrm{u}^2 + \frac{1}{\mathrm{u}^3} = 4\alpha^2 - 2, \qquad \mathrm{v}^2 + \frac{1}{\mathrm{v}^2} = 4\beta^2 - 2.$$

Il est facile de former d'après ces relations l'équation

$$X^2 - 4\alpha\beta X + 4(\alpha^2 + \beta^2) - 4 = 0;$$

dont $\mathrm{uv}+\frac{1}{\mathrm{uv}}$, $\frac{\mathrm{v}}{\mathrm{u}}+\frac{\mathrm{u}}{\mathrm{v}}$, seront les racines, et on aura, en posant

$$m=(\alpha^2-1)(\beta^2-1),$$

$$\mathrm{uv}+\frac{1}{\mathrm{uv}}=2\alpha\beta+2\sqrt{m},\quad \frac{\mathrm{u}}{\mathrm{v}}+\frac{\mathrm{v}}{\mathrm{u}}=2\alpha\beta-2\sqrt{m}.$$

Alors les termes de (13) étant tous connus en α, β, on aura

$$2(\alpha^2\beta^2+1)-(\alpha+\beta)^2=4(7+\alpha\beta)\sqrt{(\alpha^2-1)(\beta^2-1)}.$$

En élevant au carré et réduisant, il viendra

$$(14)\quad \alpha^4+\beta^4-64(\alpha^3\beta^3+\alpha\beta)-186\alpha^2\beta^2+60(\alpha\beta^3+\beta\alpha^3)+196(\alpha^2+\beta^2)-192,$$

résultat qui coïncide avec celui déjà fourni par Mr. Cayley, en suivant une autre voie.

TURIN, Juillet, 1887.

Note on Geometric Inferences from Algebraic Symmetry.

By Frank Morley.

If, on a function symmetrical in y, z, we perform an operation symmetrical in y, z, the result will be symmetrical in y, z.

From this simple principle we may infer geometric results not otherwise obvious. For instance, the equation of a quartic curve having 1 node and 2 cusps is

$$(yz + zx + xy)^2 = kx^2yz,$$

and from Plücker's equations it has 2 inflexions and 1 double tangent. The double tangent cannot pass through the node, and must therefore be, from the symmetry in y, z,

$$y + z = \alpha x.$$

The line joining the two inflexions cannot pass through the node, for the loop, being symmetrical in y, z, cannot have only one inflexion; hence this line must be symmetrical in y, z, or is

$$y + z = \beta x.$$

Again, one inflexional tangent is related to y just as the other is to z. Hence the line joining the points where they meet the curve again will be

$$y + z = \gamma x.$$

Also, from each inflexion one tangent can be drawn. The line joining the points of contact will be

$$y + z = \delta x.$$

These lines intersect on the line joining the cusps, $x = 0$.

The result, so far as concerns the double tangent and the line joining the inflexions, is stated in Salmon's Curves, §288, the proof being left to the reader. A special case is given as Exercise 143 in the Annals of Mathematics, June 1887. For another instance, take the sextic with 3 triple points, each the limit of a node and 2 cusps. Then Plücker's equations give, since $m = 6$, $\delta = 3$, $\varkappa = 6$,

$$\begin{aligned} n &= m = 6, \\ \iota &= \varkappa = 6, \\ \tau &= \delta = 3. \end{aligned}$$

The curve is

$$(yz + zx + xy)^3 + 9kx^2y^2z^2 = 0;$$

and since a double tangent cannot pass through a triple point, the 3 double tangents are

$$\begin{aligned} y + z &= ax, \\ z + x &= ay, \\ x + y &= az, \end{aligned}$$

i. e. a double tangent, the tangent at a triple point, and the line through the other triple points are concurrent. It is not easy to draw any certain inference about the 6 inflexions from the symmetry, but by forming the Hessian we can show that they lie on the conic

$$(3 - k)^2 u = 4k(x + y + z)^2,$$

writing

$$u \equiv yz + zx + xy;$$

and since all the intersections of the conic and sextic are inflexions, the conic must touch the sextic at the inflexions.

HAVERFORD COLLEGE, November 22, 1887.

Surfaces telles que l'origine se projette sur chaque normale au milieu des centres de courbure principaux.

PAR P. APPELL.

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1. Lorsque l'on cherche une surface telle qu'un pinceau infiniment délié de normales à cette surface découpe dans la surface d'une sphère de centre O, à l'entrée et à la sortie, *des aires équivalentes*, on trouve que la projection du point O sur chaque normale doit se trouver *au milieu des centres de courbure principaux* situés sur cette normale. Réciproquement, si l'on a une surface possédant cette propriété, un pinceau de normales découpera sur toute sphère de centre O, à l'entrée et à la sortie de cette sphère, des aires équivalentes. C'est ce que j'ai montré dans mon Mémoire *Sur les Déblais et Remblais*,* en intégrant l'équation différentielle de ces surfaces qui se rattachent d'une façon remarquable aux *surfaces minima*.

Je me propose actuellement de faire une étude plus approfondie de ces surfaces en donnant sous une forme simple les expressions des coordonnées d'un point d'une de ces surfaces en fonction de deux paramètres et en indiquant les équations différentielles des lignes de courbure et des lignes asymptotiques, dont les premières peuvent être intégrées dans une infinité de cas et, en particulier, pour une infinité de surfaces algébriques. J'insisterai particulièrement sur la correspondance qu'on peut établir entre nos surfaces *minima*, en démontrant qu'à toute *surface minima* on peut faire correspondre une de nos surfaces et réciproquement.

2. Considérons un système de trois axes rectangulaires Ox, Oy, Oz et une sphère S ayant pour centre l'origine O et pour rayon l'unité. Soit Σ une surface

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* Voyez *Mémoires présentés par divers savants à l'Académie des Sciences*, Tome XXIX, No. 3, pages 133-137.

non développable, α, β, γ les cosinus directeurs de la normale à la surface en un point M, ou, d'une façon plus précise, de la portion de normale comptée à partir de M dans un sens déterminé. Si par le centre O de la sphère S on mène un rayon parallèle à cette normale, cette parallèle rencontre la surface de la sphère en un point m dont les coordonnées rectangulaires sont α, β, γ et qu'on appelle *image sphérique* du point M. A chaque courbe tracée sur la surface Σ correspondra sur la sphère S une courbe qui sera appelée *image sphérique* de la première.

Supposons α, β, γ réels: comme ces trois quantités vérifient l'équation de la sphère S

$$\alpha^2 + \beta^2 + \gamma^2 = 1,$$

on pourra exprimer ces trois quantités en fonction de deux paramètres. Pour cela, nous considérons cette sphère comme une surface réglée admettant un double système de génératrices imaginaires et nous prendrons pour variables deux quantités demeurant constantes respectivement sur les génératrices de chaque système.* Pour cela remarquons que l'équation

$$\alpha^2 + \beta^2 + \gamma^2 = 1 \tag{1}$$

peut s'écrire

$$(\alpha + i\beta)(\alpha - i\beta) = (1 + \gamma)(1 - \gamma)$$

et posons

$$\left.\begin{aligned} \frac{\alpha + i\beta}{1-\gamma} &= \frac{1+\gamma}{\alpha - i\beta} = s, \\ \frac{\alpha - i\beta}{1-\gamma} &= \frac{1+\gamma}{\alpha + i\beta} = s_0, \end{aligned}\right\} \tag{2}$$

où s_0 est la quantité *imaginaire conjuguée* de s puisque α, β, γ sont supposés réels. La signification géométrique des quantités imaginaires s et s_0 est très-simple: on la trouvera dans les *Leçons* de M. G. Darboux: on pourra aussi consulter le Mémoire de M. Goursat "Sur les surfaces qui admettent les plans de symétrie d'un polyèdre régulier," Annales de l'Ecole Normale supérieure, 3$^{\text{ième}}$ série, T. IV, 1887, pages 42 et suivantes.

En résolvant les équations (2) par rapport à α, β, γ on trouve

$$\left.\begin{aligned} \alpha &= \frac{s + s_0}{1 + ss_0}, \\ \beta &= i\,\frac{s_0 - s}{1 + ss_0}, \\ \gamma &= \frac{ss_0 - 1}{1 + ss_0}. \end{aligned}\right\} \tag{3}$$

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* Voyez *Leçons sur la Théorie Générale des Surfaces* par G. Darboux (Gauthier Villars, 1887), Tome I, pages 22 et 245.

D'après la méthode employée pour la première fois par M. Bonnet* avec des variables un peu différentes, on peut définir la surface Σ en écrivant l'équation du plan tangent en M à cette surface sous la forme

$$x(s+s_0)+iy(s_0-s)+z(ss_0-1)=u, \tag{4}$$

et en regardant u comme une fonction de s et s_0,

$$u=\phi(s,s_0).$$

Lorsque cette fonction ϕ sera connue la surface Σ sera l'enveloppe des plans (4): elle sera donc entièrement connue. Les coordonnées du point de contact M du plan (4) avec la surface Σ sont données par les équations suivantes

$$\left.\begin{aligned} x+iy&=\frac{\dfrac{\partial u}{\partial s_0}+su-s^2\dfrac{\partial u}{\partial s}}{1+ss_0},\\ x-iy&=\frac{\dfrac{\partial u}{\partial s}+s_0u-s_0^2\dfrac{\partial u}{\partial s_0}}{1+ss_0},\\ z&=\frac{-u+s\dfrac{\partial u}{\partial s}+s_0\dfrac{\partial u}{\partial s_0}}{1+ss_0}\end{aligned}\right\} \tag{5}$$

Dans ce système, les équations différentielles des lignes de courbure et des lignes asymptotiques sont les suivantes:

lignes de courbure:

$$\frac{\partial^2 u}{\partial s^2}ds^2=\frac{\partial^2 u}{\partial s_0^2}ds_0^2 \tag{6}$$

lignes asymptotiques:

$$\frac{\partial^2 u}{\partial s^2}ds^2+\frac{\partial^2 u}{\partial s_0^2}ds_0^2+2dsds_0\left(\frac{\partial^2 u}{\partial s\partial s_0}+\frac{u-s\dfrac{\partial u}{\partial s}-s_0\dfrac{\partial u}{\partial s_0}}{1+ss_0}\right)=0. \tag{7}$$

Enfin les formules donnant le rayon de courbure R' et les coordonnées X', Y', Z' du centre de courbure deviennent

$$\left.\begin{aligned} 2R'&=-u+s\frac{\partial u}{\partial s}+s_0\frac{\partial u}{\partial s_0}-(1+ss_0)\left[\frac{\partial^2 u}{\partial s\partial s_0}+\sqrt{\frac{\partial^2 u}{\partial s^2}\frac{\partial^2 u}{\partial s_0^2}}\right],\\ 2Z'&=-u+s\frac{\partial u}{\partial s}+s_0\frac{\partial u}{\partial s_0}+(1-ss_0)\left[\frac{\partial^2 u}{\partial s\partial s_0}+\sqrt{\frac{\partial^2 u}{\partial s^2}\frac{\partial^2 u}{\partial s_0^2}}\right],\\ X'+Y'i&=\frac{\partial u}{\partial s_0}-s\left[\frac{\partial^2 u}{\partial s\partial s_0}+\sqrt{\frac{\partial^2 u}{\partial s^2}\frac{\partial^2 u}{\partial s_0^2}}\right],\\ X'-Y'i&=\frac{\partial u}{\partial s}-s_0\left[\frac{\partial^2 u}{\partial s\partial s_0}+\sqrt{\frac{\partial^2 u}{\partial s^2}\frac{\partial^2 u}{\partial s_0^2}}\right].\end{aligned}\right\} \tag{8}$$

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* *Journal de Liouville*, T. V, 1860.

Ces formules s'établissent aisément: on les trouvera dans les *Leçons sur la Théorie générale des surfaces* par M. G. Darboux, T. I, pages 245–246: les variables que M. Darboux appelle α, β et ξ sont nos variables s, s_0 et $-u$. Ces mêmes formules se trouvent reproduites dans le Mémoire déjà cité de M. Goursat (Annales de l'Ecole Normale, 1887, pages 46 et 47).

3. Ces formules générales étant rappelées, arrivons à l'objet de ce Mémoire, à savoir la détermination des surfaces Σ telles que *la projection de l'origine sur une normale quelconque se trouve au milieu des centres de courbure principaux.* Pour abréger le langage, nous appellerons ces surfaces les *surfaces* Σ. Voici comment on obtient leur équation différentielle. Les équations de la normale au point $M(x, y, z)$ d'une surface Σ sont

$$\frac{X-x}{\alpha}=\frac{Y-y}{\beta}=\frac{Z-z}{\gamma};$$

le plan projetant l'origine sur cette normale est

$$\alpha X+\beta Y+\gamma Z=0.$$

Donc la projection de l'origine sur la normale a pour coordonnée Z:

$$Z=z-\gamma(\alpha x+\beta y+\gamma z).$$

D'après les valeurs (3) de α, β, γ on a

$$Z=z-\frac{\gamma}{1+ss_0}\left[(s+s_0)x+i(s_0-s)y+(ss_0-1)z\right],$$

ou enfin, puisque le point de contact $M(x, y, z)$ est dans le plan tangent (4)

$$Z=z-\frac{\gamma u}{1+ss_0}.$$

D'autre part, on vérifie sans peine, d'après les formules (3) et (5), que le Z' d'un centre de courbure est lié au rayon R' correspondant par la formule (d'ailleurs évidente)

$$Z'=z+\gamma R'.$$

En appelant R'' le second rayon de courbure et Z'' sa hauteur au dessus du plan xOy, on aura de même

$$Z''=z+\gamma R''.$$

Donc

$$\frac{Z'+Z''}{2}=z+\gamma\frac{R'+R''}{2}.$$

La définition des surfaces Σ donne

$$Z = \frac{Z' + Z''}{2},$$

c'est à dire, après suppression du facteur γ:

$$\frac{R' + R''}{2} + \frac{u}{1 + ss_0} = 0,$$

équation qui signifie que la distance du plan tangent à l'origine est égale à la demi-somme des rayons de courbure principaux, propriété résultant immédiatement de la définition. En remplaçant la demi-somme des rayons de courbure principaux par son expression déduite des formules (8), on a l'équation différentielle des surfaces cherchées

$$(1 + ss_0)\frac{\partial^2 u}{\partial s \partial s_0} - s\frac{\partial u}{\partial s} - s_0\frac{\partial u}{\partial s_0} - \frac{1 - ss_0}{1 + ss_0}u = 0. \tag{9}$$

Pour intégrer cette équation, posons

$$u = v(1 + ss_0),$$

v désignant une nouvelle fonction inconnue qui n'est autre chose que la distance du plan tangent à l'origine. Par l'introduction de cette nouvelle fonction, l'équation différentielle (9) prend la forme

$$\frac{\partial^2 v}{\partial s \partial s_0} = 0,$$

qui donne immédiatement

$$v = f(s) + f_0(s_0),$$

les fonctions f et f_0 étant conjuguées. On aura donc

$$u = (1 + ss_0)[f(s) + f_0(s_0)]. \tag{10}$$

D'après les formules (5), les coordonnées d'un point de la surface seront

$$\left.\begin{aligned} x + iy &= f_0'(s_0) - s^2 f'(s) + \frac{2s}{1 + ss_0}[f(s) + f_0(s_0)], \\ x - iy &= f'(s) - s_0^2 f_0'(s_0) + \frac{2s_0}{1 + ss_0}[f(s) + f_0(s_0)], \\ z &= sf'(s) + s_0 f'(s_0) - \frac{1 - ss_0}{1 + ss_0}[f(s) + f_0(s_0)]. \end{aligned}\right\} \tag{11}$$

Ces formules résolues par rapport à x et y peuvent s'écrire

$$\left.\begin{aligned} x &= R\left[(1-s^2)f'(s) + \frac{2(s+s_0)}{1+ss_0}f(s)\right], \\ y &= R\left[i(1+s^2)f'(s) + \frac{2i(s_0-s)}{1+ss_0}f(s)\right], \\ z &= R\left[\frac{1}{2}sf'(s) - \frac{1}{2}\,\frac{1-ss_0}{1+ss_0}f(s)\right], \end{aligned}\right\} \qquad (\cancel{1}2)$$

le signe R désignant la partie réelle d'une quantité imaginaire quelconque et $f(s)$ une fonction analytique de la variable imaginaire s.

À toute fonction analytique $f(s)$ d'une variable imaginaire s correspond donc une de nos surfaces Σ qui est réelle. Lorsque l'on fait croître $f(s)$ d'une *constante*, les différentes surfaces Σ que l'on obtient sont *parallèles*, comme il résulte immédiatement de la signification géomêtrique de la quantité

$$v = f(s) + f_0(s_0) = \frac{1}{2}R\,[f(s)]$$

qui exprime la distance du plan tangent à l'origine. Lorsque la fonction analytique $f(s)$ est algébrique, la surface Σ est elle même algébrique.

Voici quelques surfaces particulières rentrant dans la catégorie des surfaces Σ.

1°). Si l'on prend $f(s) = 0$ la surface Σ correspondante est une sphère ayant pour centre l'origine.

2°). Si l'on prend

$$f(s) = a \log s,$$

a étant une constante réelle, on trouve celle des surfaces Σ qui est de révolution autour de l'axe Oz. La méridienne de cette surface est une développante d'une parabole ayant pour axe Oz et pour *foyer* l'origine O.

3°). En prenant, plus généralement

$$f(s) = (a+ib)\log s,$$

a et b étant des constantes, on aura une surface Σ possédant la propriété *d'être parallèle à elle même*, car si l'argument de s augmente de 2π, $f(s)$ augmente de la constante

$$2\pi i\,(a+ib),$$

et la nouvelle portion de surface Σ correspondante est parallèle à l'ancienne. Le cas le plus simple serait le cas de $a = 0$.

4. *Lignes de courbure.* L'équation différentielle des lignes de courbure étant

$$ds^2 \frac{\partial^2 u}{\partial s^2} = ds_0^2 \frac{\partial^2 u}{\partial s_0^2},$$

cette équation devient dans le cas actuel

$$u = (1 + ss_0)[f(s) + f_0(s_0)],$$

$$ds^2 [(1 + ss_0) f''(s) + 2s_0 f'(s)] = ds_0^2 [(1 + ss_0) f_0''(s_0) + 2s f_0'(s_0)]. \qquad (13)$$

Cette équation est intégrable si $f'(s)$ est de la forme

$$f'(s) = as^n,$$

a et n étant des *constantes* dont la seconde n est *réelle;* alors

$$f_0'(s_0) = a_0 s_0^n,$$

a_0 étant conjugué de a. En effet, dans ce cas, on peut diviser les deux membres de l'équation (13) par

$$n + (n + 2) ss_0$$

et il reste l'équation

$$as^{n-1} ds^2 = a_0 s_0^{n-1} ds_0^2,$$

dont l'intégration est immédiate. Si n est commensurable et différent de -1 les surfaces correspondantes sont *algébriques.* On peut encore effectuer l'intégration si l'on a

$$f'(s) = \frac{a}{(s + b)^2}$$

a et b étant des constantes quelconques ayant pour conjuguées a_0 et b_0, de sorte que

$$f_0'(s_0) = \frac{a_0}{(s_0 + b_0)^2}.$$

Dans ce cas l'équation (13) devient

$$\frac{a ds^2}{(b_0 s - 1)(s + b)^3} = \frac{a_0 ds_0^2}{(b s_0 - 1)(s_0 + b_0)^3},$$

où les variables sont séparées.

Lignes asymptotiques. En vertu de l'équation différentielle (9) des surfaces Σ, le coefficient de $2 ds ds_0$ dans l'équation (7) des lignes asymptotiques se réduit à

$$\frac{2u}{(1 + ss_0)^2},$$

et l'équation différentielle des lignes asymptotiques devient

$$\frac{\partial^2 u}{\partial s^2} ds^2 + \frac{\partial^2 u}{\partial s_0^2} ds_0^2 + \frac{4u}{(1 + ss_0)^2} ds ds_0 = 0,$$

ou enfin, d'après la valeur (10) de u,

$$\left.\begin{array}{l}[(1+ss_0)f''(s)+2s_0f'(s)]\,ds^2+[(1+ss_0)f_0''(s_0)+2sf_0'(s_0)]\,ds_0^2\\ \qquad +\dfrac{4[f(s)+f_0(s_0)]}{1+ss_0}\,ds\,ds_0=0.\end{array}\right\} \qquad (14)$$

5. Pour terminer cette étude nous allons montrer comment à chaque surface Σ se rattache une surface minima et réciproquement. L'équation différentielle des surfaces minima est

$$(1+ss_0)\frac{\partial^2 u_1}{\partial s\partial s_0}-s\frac{\partial u_1}{\partial s}-s_0\frac{\partial u_1}{\partial s_0}+u_1=0, \qquad (15)$$

et celle des surfaces Σ

$$(1+ss_0)\frac{\partial^2 u}{\partial s\partial s_0}-s\frac{\partial u}{\partial s}-s_0\frac{\partial u}{\partial s_0}-\frac{1-ss_0}{1+ss_0}u=0. \qquad (9)$$

En différentiant l'équation (15) par rapport à l'une quelconque des variables, on obtient les deux équations

$$\left.\begin{array}{l}(1+ss_0)\dfrac{\partial^3 u_1}{\partial s^2\partial s_0}-s\dfrac{\partial^2 u_1}{\partial s^2}=0,\\ (1+ss_0)\dfrac{\partial^3 u_1}{\partial s\partial s_0^2}-s_0\dfrac{\partial^2 u_1}{\partial s_0^2}=0.\end{array}\right\} \qquad (16)$$

Ceci posé si l'on a une solution u_1 de l'équation différentielle (15) des surfaces minima, on en déduira une solution u de l'équation (9) des surfaces Σ en faisant

$$u=u_1-s\frac{\partial u_1}{\partial s}-s_0\frac{\partial u_1}{\partial s_0}, \qquad (17)$$

comme on le vérifie immédiatement à l'aide des relations (15) et (16).

Réciproquement, soit u une solution de l'équation (9), il existera une fonction u_1 vérifiant les équations (15) et (17). En effet l'équation (17) donne par la différentiation

$$\frac{\partial u}{\partial s}=-s\frac{\partial^2 u_1}{\partial s^2}-s_0\frac{\partial^2 u_1}{\partial s\partial s_0},$$

$$\frac{\partial u}{\partial s_0}=-s\frac{\partial^2 u_1}{\partial s\partial s_0}-s_0\frac{\partial^2 u_1}{\partial s_0^2},$$

et comme (15) donne

$$u=-(1+ss_0)\frac{\partial^2 u_1}{\partial s\partial s_0},$$

on a, en définitive,

$$\frac{\partial^2 u_1}{\partial s \partial s_0} = -\frac{u}{(1+ss_0)},$$

$$\frac{\partial^2 u_1}{\partial s^2} = -\frac{1}{s}\frac{\partial u}{\partial s} + \frac{s_0}{s}\frac{u}{(1+ss_0)},$$

$$\frac{\partial^2 u_1}{\partial s_0^2} = -\frac{1}{s_0}\frac{\partial u}{\partial s_0} + \frac{s}{s_0}\frac{u}{(1+ss_0)}.$$

La fonction u étant supposée vérifier l'équation (9), la quantité

$$d\frac{\partial u_1}{\partial s} = -\frac{u}{1+ss_0}ds_0 - \left[\frac{1}{s}\frac{\partial u}{\partial s} - \frac{s_0}{s}\frac{u}{1+ss_0}\right]ds$$

est une différentielle exacte. L'intégration de cette différentielle donnera $\frac{\partial u_1}{\partial s}$,

$$\frac{\partial u_1}{\partial s} = \phi(s, s_0) + c;$$

de même

$$-\frac{u}{(1+ss_0)}ds - \left[\frac{1}{s_0}\frac{\partial u}{\partial s_0} - \frac{s}{s_0}\frac{u}{1+ss_0}\right]ds_0$$

est une différentielle exacte dont l'intégration donnera

$$\frac{\partial u_1}{\partial s_0} = \psi(s, s_0) + c_0$$

et alors l'équation (17) donnera enfin

$$u_1 = u + s[\phi(s, s_0) + c] + s_0[\psi(s, s_0) + c_0]$$

avec deux constantes arbitraires c et c_0 imaginaires conjuguées. À chaque fonction u correspondent donc une *infinité* de fonctions u_1 avec deux constantes c et c_0; mais les surfaces minima que l'on obtient en faisant varier les constantes c et c_0 se déduisent de l'une d'entre elles par une translation parallèle au plan xOy. Réciproquement on vérifie immédiatement que si l'on transporte une surface minima parallèlement au plan xOy en changeant la fonction u_1 relative à cette surface en

$$u_1 + cs + c_0 s_0,$$

la fonction u donnée par l'équation (17) ne change pas et par suite la surface Σ correspondant à la surface minima ne change pas.

Donc, en regardant comme identiques les surfaces minima qui se déduisent l'une de l'autre par une translation parallèle au plan des xy, à toute surface minima *correspond point par point une seule surface Σ et réciproquement: les plans tangents aux deux surfaces aux points correspondants sont parallèles.*

Nos formules permettent encore de montrer que, si l'on transporte une surface minima parallèlement à l'axe Oz, les surfaces Σ correspondantes ~~sont~~ parallèles; et réciproquement. Cela résulte de ce que, si l'on ajoute à u_1 la quantité

$$c\,(1 - ss_0)$$

la fonction u donnée par la formule (17) augmente de

$$c\,(1 + ss_0).$$

Ces résultats deviennent en partie évidents par l'interprétation géométrique de la formule (17)

$$u = u_1 - s\,\frac{\partial u_1}{\partial s} - s_0\,\frac{\partial u_1}{\partial s_0}. \tag{17}$$

Si l'on appelle z_1 la coordonnée z d'un point de la surface minima, on a d'après (5)

$$z_1 = \frac{-u_1 + s\,\dfrac{\partial u_1}{\partial s} + s_0\,\dfrac{\partial u_1}{\partial s_0}}{1 + ss_0}.$$

La formule de transformation (17) peut donc s'écrire

$$u\,(1 + ss_0) = -z_1,$$

ou, en appelant, comme plus haut, v la distance au point O du plan tangent à la surface Σ:

$$v = -z_1.$$

On a donc la construction suivante: *Soit une surface S_1 et un plan tangent P_1 à cette surface au point M_1. Construisons un plan parallèle P situé à une distance de l'origine O égale à la distance du point M_1 au plan xOy. Lorsque le point M_1 décrira S_1, ce plan P enveloppera la surface transformée S_2.*

En particulier, si le point M_1 décrit une *surface minima* le plan P enveloppera une de nos surfaces Σ. Inversement, le calcul nous a montré qu'une surface Σ donnée ne peut être déduite par ce procédé que d'une surface minima.

6. M. Bonnet a indiqué* une transformation qui permet de déduire de chaque surface minima une surface telle que *le milieu des deux centres de courbure principaux situés sur une normale se trouve dans un plan fixe xOy.* J'ai montré,

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* Comptes rendus de l'Académie des Sciences de Paris, T. XLII, page 486. Cette transformation est rappelée dans l'Ouvrage déjà cité de M. Darboux, à la page 255.

dans mon Mémoire sur les *Déblais et Remblais* précédemment cité qu'un pinceau de normales à une des surfaces de M. Bonnet découpe des aires équivalentes sur deux plans parallèles au plan xOy et équidistants de ce plan.

Les surfaces Σ que nous venons d'étudier se déduisent d'une façon fort simple des surfaces de M. Bonnet dont nous avons rappelé à l'instant la définition. En effet d'après la seconde des équations (8) qui détermine les coordonnées Z des deux centres de courbures principaux, l'équation différentielle des surfaces en question est

$$(1-ss_0)\frac{\partial^2 u_2}{\partial s\,\partial s_0}+s\frac{\partial u_2}{\partial s}+s_0\frac{\partial u_2}{\partial s_0}-u_2=0. \tag{18}$$

Soit u_2 une solution de cette équation, la fonction

$$v=\frac{-s\dfrac{\partial u_2}{\partial s}-s_0\dfrac{\partial u_2}{\partial s_0}+u_2}{1-ss_0}=\frac{\partial^2 u_2}{\partial s\,\partial s_0},$$

vérifie l'équation
$$\frac{\partial^2 v}{\partial s\,\partial s_0}=0,$$

comme on s'en assure immédiatement en différentiant l'équation (18) successivement par rapport à s et à s_0. Donc la fonction

$$u=(1+ss_0)v=\frac{1+ss_0}{1-ss_0}\left(u_2-s\frac{\partial u_2}{\partial s}-s_0\frac{\partial u_2}{\partial s_0}\right)$$

vérifie l'équation différentielle (9) de nos surfaces Σ.

La correspondance entre la surface Σ et la surface S_2 vérifiant l'équation (18) est la suivante. Appelons z_2 le z d'un point M_2 de la surface S_2,

$$z_2=\frac{-u_2+s\dfrac{\partial u_2}{\partial s}+s_0\dfrac{\partial u_2}{\partial s_0}}{1+ss_0};\ (\text{éq. }5).$$

On a
$$v=\frac{ss_0+1}{ss_0-1}z_2,$$

ce qui signifie que v est égal à la longueur de la normale à la surface S_2 comprise entre le pied M_2 de cette normale et le plan xOy. Comme v désigne la distance à l'origine du plan tangent à la surface Σ, on a la construction suivante:

Etant donnée une surface S_2 de M. Bonnet, on mène en un point M_2 de cette surface le plan tangent P_2 et la normale M_2N_2 jusqu'au plan xOy; le plan P parallèle à P_2 et situé à une distance de l'origine égale à la normale M_2N_2 enveloppe une surface Σ.

On pourrait montrer qu'a toute surface Σ répond inversement une surface S_2: il suffirait pour cela de refaire un calcul analogue à celui de la page 182.

Cette correspondance entre les surfaces Σ et S_2 montre que:

De tout système de routes servant à déblayer une aire plane homogène sur une aire équivalente située dans un plan parallèle, on peut déduire un système de routes servant à déblayer une aire sphérique homogène sur une aire équivalente située sur la même sphère.

Les routes servant au premier déblai seront normales à une surface S_2, et les routes servant au second déblai normales à une surface Σ.

Surfaces telles que la somme des rayons de courbure principaux est proportionnelle à la distance d'un point fixe au plan tangent.

PAR E. GOURSAT.

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1. Dans un Mémoire récent, publié dans l'*American Journal of Mathematics*, Vol. X, No. 2, p. 175, M. Appell a étudié les surfaces telles qu'un point fixe se projette sur chaque normale au milieu des centres de courbure principaux. J'étudie dans ce travail des surfaces jouissant d'une propriété un peu plus générale; la détermination de ces surfaces dépend de l'intégration d'une équation linéaire aux dérivées partielles, qui peut être intégrée sous forme explicite par la méthode de Laplace dans un nombre illimité de cas, dont les plus simples fournissent précisément les surfaces minima et les surfaces étudiées par M. Appell. De chaque surface de cette espèce on peut en déduire une nouvelle par une construction géométrique, qui comprend comme cas particulier la construction donnée par M. Appell.

Je montre en terminant comment on peut ramener à un problème résolu par Riemann la recherche des surfaces de cette nature tangentes à une développable donnée le long d'une courbe donnée.

2. Considérons un système de trois axes rectangulaires Ox, Oy, Oz et une sphère S de rayon égal à l'unité ayant pour centre l'origine. Soit Σ une surface non développable, M un point de cette surface, a, b, c les cosinus directeurs d'une direction déterminée MN sur la normale à la surface Σ au point M. Si par l'origine on mène une parallèle à cette direction, cette droite rencontre la sphère en un point bien déterminé m, dont les coordonnées rectangulaires sont a, b, c, qui est dit l'*image sphérique* du point M. A chaque courbe tracée sur Σ correspond ainsi une courbe tracée sur la sphère qui sera appelée *image sphérique* de la première.

Pour représenter la position du point m de la sphère, on exprime les coordonnées a, b, c en fonction de deux paramètres. La sphère S

$$a^2+b^2+c^2=1$$

pouvant être considérée comme une surface réglée, on sait que par chaque point passent deux génératrices rectilignes imaginaires. Nous prendrons comme variables indépendantes deux quantités restant constantes respectivement sur les génératrices de chaque système. L'équation de la sphère pouvant s'écrire

$$(a+ib)(a-ib)=(1+c)(1-c), \tag{1}$$

nous avons immédiatement les deux systèmes de génératrices

$$\left.\begin{aligned} \frac{a+ib}{1-c}&=\frac{1+c}{a-ib}=s,\\ \frac{a-ib}{1-c}&=\frac{1+c}{a+ib}=s_0, \end{aligned}\right\} \tag{2}$$

s_0 désignant la quantité imaginaire conjuguée de s lorsque a, b, c sont réels. La signification géométrique de ces quantités s, s_0 est bien connue ; si a, b, c sont réels et si on fait la projection stéréographique du point m de la sphère sur le plan des xy, le point de vue étant le point de la sphère situé sur la partie positive de l'axe Oz, la quantité imaginaire s est l'*affixe* de la projection. Des équations (2) on tire inversement

$$\left.\begin{aligned} a&=\frac{s+s_0}{1+ss_0},\\ b&=i\,\frac{s_0-s}{1+ss_0},\\ c&=\frac{ss_0-1}{1+ss_0}. \end{aligned}\right\} \tag{3}$$

L'équation du plan tangent à la surface Σ au point M pourra s'écrire

$$x(s+s_0)+iy(s_0-s)+z(ss_0-1)=u, \tag{4}$$

en posant, pour abréger, $u=(1+ss_0)\delta$, δ désignant la distance de l'origine à ce plan. Si dans cette équation on regarde u comme une fonction donnée de s et de s_0, le plan représenté par cette équation enveloppe une certaine surface et on peut regarder la relation

$$u=\Phi(s,s_0)$$

comme l'équation d'une surface dans le système de variables adopté. Les coordonnées du point de contact M du plan (4) avec la surface Σ sont données par les équations suivantes

$$\left.\begin{aligned} x+iy &= \frac{\dfrac{\partial u}{\partial s_0}+su-s^2\dfrac{\partial u}{\partial s}}{1+ss_0}, \\ x-iy &= \frac{\dfrac{\partial u}{\partial s}+s_0u-s_0^2\dfrac{\partial u}{\partial s_0}}{1+ss_0}, \\ z &= \frac{s\dfrac{\partial u}{\partial s}+s_0\dfrac{\partial u}{\partial s_0}-u}{1+ss_0}. \end{aligned}\right\} \qquad (5)$$

Dans ce système, les équations différentielles des lignes de courbure et des lignes asymptotiques sont les suivantes :

lignes de courbure :

$$\frac{\partial^2 u}{\partial s^2}ds^2=\frac{\partial^2 u}{\partial s_0^2}ds_0^2; \qquad (6)$$

lignes asymptotiques :

$$\frac{\partial^2 u}{\partial s^2}ds^2+\frac{\partial^2 u}{\partial s_0^2}ds_0^2+2dsds_0\left[\frac{\partial^2 u}{\partial s\partial s_0}+\frac{u-s\dfrac{\partial u}{\partial s}-s_0\dfrac{\partial u}{\partial s_0}}{1+ss_0}\right]=0. \qquad (7)$$

Les formules donnant le rayon de courbure principal R' et les coordonnées X', Y', Z' du centre de courbure deviennent :

$$\left.\begin{aligned} 2R' &= -u+s\frac{\partial u}{\partial s}+s_0\frac{\partial u}{\partial s_0}-(1+ss_0)\left[\frac{\partial^2 u}{\partial s\partial s_0}+\sqrt{\frac{\partial^2 u}{\partial s^2}\frac{\partial^2 u}{\partial s_0^2}}\right], \\ 2Z' &= -u+s\frac{\partial u}{\partial s}+s_0\frac{\partial u}{\partial s_0}+(1-ss_0)\left[\frac{\partial^2 u}{\partial s\partial s_0}+\sqrt{\frac{\partial^2 u}{\partial s^2}\frac{\partial^2 u}{\partial s_0^2}}\right], \\ X'+Y'i &= \frac{\partial u}{\partial s_0}-s\left[\frac{\partial^2 u}{\partial s\partial s_0}+\sqrt{\frac{\partial^2 u}{\partial s^2}\frac{\partial^2 u}{\partial s_0^2}}\right], \\ X'-iY' &= \frac{\partial u}{\partial s}-s_0\left[\frac{\partial^2 u}{\partial s\partial s_0}+\sqrt{\frac{\partial^2 u}{\partial s^2}\frac{\partial^2 u}{\partial s_0^2}}\right]. \end{aligned}\right\} \qquad (8)$$

On trouvera ces formules dans les *Leçons sur la théorie générale des surfaces* de M. G. Darboux, t. I, pages 245–246 ; les variables s, s_0, $-u$ sont appelées par M. Darboux α, β, ξ. Ces formules se trouvent aussi reproduites au début du Mémoire déjà cité de M. Appell.

Etant données plusieurs surfaces Σ, Σ', Σ'', ... considérons les points m, m', m'', ... de ces surfaces où les plans tangents sont parallèles et la résultante géométrique OM des droites om, om', om'', ... Le point M décrit une surface qui est dite la résultante géométrique des surfaces Σ, Σ', Σ'', ... Si u, u', u'', ... sont les fonctions de s et de s_0 qui définissent respectivement les surfaces Σ, Σ', Σ'', ... la fonction U qui fournit la surface résultante sera

$$U = u + u' + u'' + \ldots .$$

La somme des rayons de courbure principaux $R' + R''$ étant une fonction linéaire de u et de ses dérivées $\frac{\partial u}{\partial s}$, $\frac{\partial u}{\partial s_0}$, $\frac{\partial^2 u}{\partial s \partial s_0}$, on voit immédiatement que la somme des rayons de courbure principaux de la surface résultante est égale à la somme des rayons de courbure principaux de toutes les surfaces composantes aux points correspondants. En particulier, si on a comme surfaces composantes une sphère et une surface minima, la surface obtenue, qui sera parallèle à une surface minima, jouira de cette propriété que la somme des rayons de courbure principaux sera constante, et inversement toute surface possédant cette propriété sera parallèle à une surface minima.

3. J'arrive maintenant à l'objet de ce Mémoire, qui est d'étudier les surfaces telles que *la somme des rayons de courbure principaux est proportionnelle à la distance d'un point fixe au plan tangent.* Supposons que nous ayons pris ce point fixe pour origine. La distance de l'origine au plan tangent est égale, nous l'avons vu, au signe près, à

$$\frac{u}{1 + s s_0};$$

si nous appelons X_1, Y_1, Z_1 les coordonnées du point milieu des centres de courbure principaux, on a

$$X_1 + iY_1 = \frac{\partial u}{\partial s_0} - s\frac{\partial^2 u}{\partial s \partial s_0},$$

$$X_1 - iY_1 = \frac{\partial u}{\partial s} - s_0\frac{\partial^2 u}{\partial s \partial s_0},$$

$$Z_1 = \frac{1 - s s_0}{2}\frac{\partial^2 u}{\partial s \partial s_0} + \frac{1}{2}\left(s\frac{\partial u}{\partial s} + s_0\frac{\partial u}{\partial s_0} - u\right),$$

et la distance de ce point au plan tangent est égale, au signe près, à

$$\frac{1}{2}\left[(1 + s s_0)\frac{\partial^2 u}{\partial s \partial s_0} - s\frac{\partial u}{\partial s} - s_0\frac{\partial u}{\partial s_0} + u\right],$$

c'est-à-dire à la demi-somme des rayons de courbure principaux, comme il était évident *a priori.* L'équation différentielle des surfaces cherchées sera par conséquent

$$(1+ss_0)\frac{\partial^2 u}{\partial s\partial s_0}-s\frac{\partial u}{\partial s}-s_0\frac{\partial u}{\partial s_0}+u\left[1+\frac{2K}{1+ss_0}\right]=0. \tag{9}$$

Il est aisé de voir que la constante K représente le rapport des distances du point milieu des centres de courbure principaux et de l'origine au plan tangent, ce rapport étant pris positivement lorsque l'origine et le point milieu des centres de courbure principaux sont de côtés différents du plan tangent.

Pour $K=0$, l'équation (9) se réduit à l'équation des surfaces minima; pour $K=-1$, on retrouve l'équation différentielle des surfaces étudiées par M. Appell. Dans ces deux cas, l'intégrale générale de l'équation (9) peut être obtenue sous forme explicite; on peut l'obtenir en particulier par l'application de la méthode de Laplace. Nous allons voir qu'il existe une infinité de valeurs de K pour lesquelles l'application de cette méthode fournit sous forme explicite l'intégrale générale de l'équation (9).

De la forme linéaire de l'équation (9) on conclut que, si l'on a plusieurs intégrales, leur somme sera aussi une intégrale. En d'autres termes, si on a plusieurs surfaces répondant à la question, leur surface résultante jouira de la même propriété. C'est, comme on voit, l'extension à nos surfaces d'une propriété bien connue des surfaces minima. Il serait d'ailleurs facile de l'établir géométriquement d'après ce qui a été dit plus haut sur les surfaces résultantes.

4. Dans l'équation (9) faisons le changement de variables

$$s=\alpha,\quad s_0=-\frac{1}{\beta};$$

on aura

$$\frac{\partial u}{\partial s}=\frac{\partial u}{\partial \alpha},\quad \frac{\partial u}{\partial s_0}=\frac{\partial u}{\partial \beta}\beta^2,\quad \frac{\partial^2 u}{\partial s\partial s_0}=\frac{\partial^2 u}{\partial\alpha\partial\beta}\beta^2,$$

et l'équation devient

$$\beta(\beta-\alpha)\frac{\partial^2 u}{\partial\alpha\partial\beta}-\alpha\frac{\partial u}{\partial\alpha}+\beta\frac{\partial u}{\partial\beta}+u\frac{\beta(1+2K)-\alpha}{\beta-\alpha}.$$

Posons ensuite $u=\frac{\xi}{\beta}$; on trouve la nouvelle équation

$$(\alpha-\beta)\frac{\partial^2\xi}{\partial\alpha\partial\beta}+\frac{\partial\xi}{\partial\alpha}-\frac{\partial\xi}{\partial\beta}+\frac{2K\xi}{\alpha-\beta}=0; \tag{10}$$

du reste, on obtiendrait immédiatement cette équation en employant le premier système de variables α, β, ξ employé par M. Darboux (*Leçons sur la théorie générale des surfaces*, t. I, pages 244–245). Posons encore

$$\xi=(\alpha-\beta)^\lambda v;$$

l'équation (10) devient

$$(\alpha-\beta)\frac{\partial^2 v}{\partial\alpha\partial\beta}-(\lambda-1)\frac{\partial v}{\partial\alpha}+(\lambda-1)\frac{\partial v}{\partial\beta}+\frac{3\lambda-\lambda^2+2K}{\alpha-\beta}v=0.$$

Choisissons pour λ une racine de l'équation

$$\lambda^2-3\lambda-2K=0 \tag{11}$$

et posons $\lambda=1+m$; la nouvelle équation prend la forme très-simple

$$(\alpha-\beta)\frac{\partial^2 v}{\partial\alpha\partial\beta}-m\frac{\partial v}{\partial\alpha}+m\frac{\partial v}{\partial\beta}=0. \tag{12}$$

En réunissant les trois transformations précédentes, on voit qu'on passe de l'équation (9) à l'équation (12) en posant:

$$s=\alpha,\ s_0=-\frac{1}{\beta},\ u=\frac{(\alpha-\beta)^\lambda}{\beta}v, \tag{13}$$

où λ désigne une racine de l'équation (11) et où $\lambda=1+m$; la constante K est donnée en fonction de m par la formule

$$K=\frac{(m+1)(m-2)}{2}. \tag{14}$$

Inversement, si on connaît une intégrale de l'équation (12), les formules (13) permettront d'en déduire une fonction u de s et de s_0 vérifiant l'équation (9). En général cette fonction u ainsi obtenue ne prendra pas de valeurs réelles lorsque les variables s et s_0 prendront des valeurs imaginaires conjuguées et, par conséquent, ne fournira pas une surface réelle. Mais il est facile d'éviter cet inconvénient. Soit, en effet,

$$u=f(s,s_0)$$

une première intégrale de l'équation (9); comme cette équation ne change pas quand on y échange les variables s, s_0 et que tous les coefficients sont réels, la fonction $f_0(s_0,s)$, où f_0 désigne la fonction conjuguée de f, sera aussi une intégrale de la même équation. Il en sera encore de même de la somme

$$f(s,s_0)+f_0(s_0,s),$$

et il est clair que cette dernière fonction est réelle lorsque s et s_0 prennent des valeurs imaginaires conjuguées.

5. L'équation (12), à laquelle nous sommes conduits, s'est déjà présentée, sous des formes un peu différentes, dans un grand nombre de recherches d'Analyse ou de Physique Mathématique. Etudiée d'abord par Euler dans le tome III de

son *Calcul intégral,* puis généralisée par Laplace,[*] elle a été l'objet de travaux très importants de Poisson,[†] de Riemann,[‡] et plus récemment de M. Darboux,[§] qui l'a étudiée en détail, ainsi qu'une équation plus générale, dans ses leçons de la Faculté des Sciences de Paris pendant le semestre d'hiver 1887–1888. Je ne me servirai ici que des propriétés les plus simples de cette équation, en indiquant comment on peut les établir.

Lorsque m est quelconque, on ne peut pas obtenir pour l'intégrale générale de l'équation (12) une formule générale où figurent explicitement les fonctions arbitraires et leurs dérivées jusqu'à un ordre déterminé; dans le Mémoire déjà cité, Poisson a donné une forme générale de l'intégrale qui contient deux fonctions arbitraires sous des signes d'intégration définie. Mais on peut toujours obtenir, quelle que soit la constante m, une infinité d'intégrales particulières. Ainsi, en cherchant les solutions de l'équation (12) qui sont homogènes en β et α, on est conduit à l'équation différentielle linéaire à laquelle satisfait la série hypergéométrique, et on trouve ainsi que les fonctions

$$\left.\begin{aligned} v &= \alpha^{\mu} F\left(-\mu,\ m,\ 1-m-\mu,\ \frac{\beta}{\alpha}\right), \\ v &= \alpha^{-m}\beta^{m+\mu} F\left(m,\ 2m+\mu,\ 1+m+\mu,\ \frac{\beta}{\alpha}\right), \end{aligned}\right\} \tag{15}$$

où F désigne la série hypergéométrique de Gauss, vérifient, pour toute valeur de μ, l'équation (12). Pour avoir des solutions entières, il suffira de prendre pour μ un nombre entier positif. De même, en cherchant si l'équation (12) admet des intégrales qui soient le produit d'une fonction de α par une fonction de β, on trouve que la fonction

$$v = (\alpha - h)^{-m}(\beta - h)^{-m} \tag{16}$$

satisfait, pour toute valeur de h, à l'équation (12). Enfin on vérifie sans difficulté que, si $\phi(\alpha, \beta)$ est une intégrale, il en sera encore de même de la fonction

$$(a\alpha + b)^{-m}(a\beta + b)^{-m}\phi\left(\frac{c\alpha + d}{a\alpha + b},\ \frac{c\beta + d}{a\beta + d}\right), \tag{17}$$

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[*] *Recherches sur le calcul intégral aux différences partielles*, par M. DE LA PLACE. *Mémoires de Mathématique et de Physique de l'Académie des Sciences pour* 1773, p. 341–403.

[†] POISSON, *Mémoire sur l'intégration des équations linéaires aux dérivées partielles.* (*Journal de l'Ecole Polytechnique,* t. XII, XIX$^{\text{ème}}$ Cahier, p. 215; 1823.)

[‡] RIEMANN, *Ueber die Fortpflanzung ebener Luftwellen von endlicher Schwingungsweite.* (*Gesammelte Werke,* p. 145.)

[§] DARBOUX, *Sur une équation lineaire aux dérivées partielles.* (*Comptes rendus,* t. XCV, p. 69; juillet 1880.)

quelles que soient les constantes a, b, c, d. On voit donc qu'on pourra toujours obtenir une infinité de surfaces réelles répondant à la question, quelle que soit la valeur de la constante K, et même de surfaces algébriques pourvu que la valeur de λ fournie par l'équation (11) soit commensurable.

Pour abréger je désignerai dans la suite par $E(m)$ l'équation (12) et par Σ_m une quelconque de nos surfaces correspondant à cette valeur de m. Comme à chaque valeur de K l'équation (14) fait correspondre deux valeurs de m dont la somme est égale à l'unité, on voit que les surfaces Σ_m et Σ_{1-m} sont identiques. D'ailleurs on passe immédiatement de l'équation $E(m)$ à l'équation $E(1-m)$ en multipliant les intégrales par

$$(\alpha-\beta)^{2m-1}.$$

Si on désigne, d'une manière générale, par V_m une intégrale quelconque de l'équation $E(m)$, on peut écrire

$$V_{1-m}=(\alpha-\beta)^{2m-1}V_m. \tag{18}$$

Je dirai que deux équations $E(m)$ et $E(m')$ sont *contigües* quand elles correspondent à des valeurs de m qui diffèrent d'une unité; les surfaces Σ correspondantes seront dites aussi contigües. A une série de surfaces Σ_m ou Σ_{1-m} correspondent deux séries de surfaces contigües Σ_{m+1} ou Σ_{-m}, et Σ_{m-1} ou Σ_{2-m}. Je supposerai dans ce qui suit que les valeurs de m, racines de l'équation (14), sont réelles, c'est-à-dire que la constante K est supérieure à $-\frac{9}{8}$. On pourra même supposer, si l'on veut, que la valeur de m est supérieure à $\frac{1}{2}$.

6. Laissant de côté ces généralités, je considère maintenant le cas où l'équation (12) peut être intégrée par la méthode de Laplace; pour qu'il en soit ainsi, *il faut et il suffit que m soit un nombre entier*. On a donc une suite illimitée de cas d'intégrabilité. Puisque les valeurs m et $1-m$ ne donnent pas des surfaces différentes. on pourrait se borner à prendre les valeurs positives de m; mais il vaut mieux, pour la suite, considérer la suite des valeurs entières, tant négatives que positives, de m avec la suite des valeurs correspondantes de K

m....	-5,	-4,	-3,	-2,	-1,	0,	1,	2,	3,	4,	5,	6,	....,
K....	14,	9,	5,	2,	0,	-1,	-1,	0,	2,	5,	9,	14,	....

Chaque valeur de K se présente deux fois dans cette suite. Ainsi pour $m=0$ et $m=1$, on a $K=-1$, et on obtient les surfaces étudiées par M. Appell dans le travail cité plus haut. Pour $m=2$, et $m=-1$, on retrouve les surfaces minima. Pour $m=3$, et $m=-2$, on a $K=2$; les surfaces obtenues sont telles que la somme des rayons de courbure principaux est égale à quatre fois la distance de l'origine au plan tangent, et ainsi de suite.

Il nous reste à montrer comment on peut dans ce cas obtenir effectivement l'intégrale générale de l'équation (12) et par suite de l'équation (9). On peut évidemment supposer pour cela que m est un entier positif. En différentiant le premier membre de l'équation (12) par rapport à α et à β successivement, et en posant

$$W=\frac{\partial^2 v}{\partial\alpha\partial\beta},$$

on trouve que W vérifie l'équation

$$(\alpha-\beta)\frac{\partial^2 W}{\partial\alpha\partial\beta}-(m+1)\frac{\partial W}{\partial\alpha}+(m+1)\frac{\partial W}{\partial\beta}=0, \tag{19}$$

qui n'est autre que l'équation $E(m+1)$. Si donc V_m désigne une intégrale de l'équation $E(m)$, la fonction

$$V_{m+1}=\frac{\partial^2 V_m}{\partial\alpha\partial\beta}, \tag{20}$$

sera une intégrale de l'équation $E(m+1)$. Mais il n'en résulte pas qu'en prenant pour V_m l'intégrale générale de l'équation $E(m)$ on obtienne de cette façon l'intégrale générale de l'équation $E(m+1)$. Pour examiner ce point essentiel, je considère l'équation intermédiaire qui est satisfaite par la fonction

$$v_1=\frac{\partial v}{\partial\alpha}, \qquad (\alpha-\beta)\frac{\partial^2 v_1}{\partial\alpha\partial\beta}-m\frac{\partial v_1}{\partial\alpha}+(m+1)\frac{\partial v_1}{\partial\beta}=0. \tag{21}$$

Soit v_1 une intégrale quelconque de l'équation (21); cherchons s'il existe une intégrale de l'équation (12) telle que l'on ait

$$\frac{\partial v}{\partial\alpha}=v_1.$$

L'équation (12) peut alors s'écrire

$$(\alpha-\beta)\frac{\partial v_1}{\partial\beta}-mv_1+m\frac{\partial v}{\partial\beta}=0,$$

et on en tirera $\frac{\partial v}{\partial\beta}$, pourvu que m ne soit pas nul. Les valeurs de $\frac{\partial v}{\partial\alpha}$ et de $\frac{\partial v}{\partial\beta}$ ainsi obtenues satisfont à la condition d'intégrabilité, d'après l'équation (21). Par conséquent, si m n'est pas nul, on obtient l'intégrale générale de l'équation

(21) en prenant $v_1 = \frac{\partial v}{\partial \alpha}$, v désignant l'intégrale générale de l'équation (12). On démontrera de la même manière que la formule

$$W = \frac{\partial v_1}{\partial \beta},$$

où v_1 désigne l'intégrale générale de l'équation (21), représente l'intégrale générale de l'équation (19), pourvu que m ne soit pas nul. Ainsi, tant que m est différent de zéro, la formule (20) permet de déduire l'intégrale générale de l'équation $E(m+1)$ de l'intégrale générale de l'équation $E(m)$.

Si nous supposons que m soit un nombre entier positif, l'application répétée de la formule (20) nous donne pour l'intégrale générale de l'équation $E(m)$

$$V_m = \frac{\partial^{2m-2} V_1}{\partial \alpha^{m-1} \partial \beta^{m-1}}, \tag{22}$$

V_1 désignant l'intégrale générale de l'équation $E(1)$. Or cette équation $E(1)$ s'intégre immédiatement, car on peut l'écrire

$$\frac{\partial^2 [(\alpha - \beta) v]}{\partial \alpha \partial \beta} = 0;$$

on en tire

$$V_1 = \frac{f(\alpha) - \varphi(\beta)}{\alpha - \beta},$$

$f(\alpha)$ désignant une fonction quelconque de α et $\phi(\beta)$ une fonction quelconque de β, et par suite

$$V_m = \frac{\partial^{2m-2} \left[\frac{f(\alpha) - \varphi(\beta)}{\alpha - \beta} \right]}{\partial \alpha^{m-1} \partial \beta^{m-1}}. \tag{23}$$

Si dans cette formule on fait maintenant le changement de variables (13), on obtiendra, pour représenter l'intégrale générale de l'équation (9), une formule où les variables s, s_0 n'entreront pas symétriquement. Pour éviter cet inconvénient, on pourra opérer comme il suit. Dans la formule générale (23) prenons la partie qui contient la fonction arbitraire de α

$$V_m = \frac{\partial^{2m-2} \left[\frac{f(\alpha)}{\alpha - \beta} \right]}{\partial \alpha^{m-1} \partial \beta^{m-1}},$$

et faisons dans cette partie le changement de variables. Nous obtenons ainsi une intégrale de l'équation (9) de la forme

$$u = A f^{m-1}(s) + B f^{m-2}(s) + \dots + L f(s), \tag{24}$$

$A, B, C, \ldots . L$ désignant des fonctions déterminées de s et de s_0, $f(s)$ une fonction arbitraire de s, et $f'(s), \ldots . f^{m-1}(s)$ ses dérivées. Désignons par $A_0, B_0, C_0, \ldots . L_0$ ce que deviennent les fonctions $A, B, C, \ldots . L$ quand on y permute les lettres s et s_0, par $f_0(s)$ une fonction arbitraire de s_0. L'intégrale générale de l'équation (9) sera alors

$$\left.\begin{aligned} u = {} & A f^{m-1}(s) + B f^{m-2}(s) + \ldots . + L f(s) \\ & + A_0 f_0^{m-1}(s_0) + B_0 f_0^{m-2}(s_0) + \ldots . + L_0 f_0(s_0), \end{aligned}\right\} \tag{25}$$

et, pour obtenir des surfaces réelles, il suffira de prendre pour f et f_0 des fonctions conjuguées. Si on porte ensuite cette valeur de u dans les formules (5), on aura les coordonnées d'un point de la surface exprimées en fonction des deux paramètres variables s, s_0.

Appliquons cette méthode aux cas les plus simples:

$K = -1$, $m = 1$, $m = 0$, *Surfaces* Σ_0 *ou* Σ_1 *de M. Appell*:

$$u = (1 + s s_0)[f(s) + f_0(s_0)];$$

$K = 0$, $m = 2$, $m = -1$, *Surfaces minima ou surfaces* Σ_2 *et* Σ_{-1}:

$$u = (1 + s s_0)[f'(s) + f_0'(s_0)] - 2 s_0 f(s) - 2 s f_0(s_0);$$

$K = 2$, $m = 3$, $m = -2$, *Surfaces* Σ_3 *ou* Σ_{-2}:

$$u = (1 + s s_0)[f''(s) + f_0''(s_0)] - 6\,[s f_0'(s_0) + s_0 f'(s)] + 12\,\frac{s^2 f_0(s_0) + s_0^2 f(s)}{1 + s s_0}.$$

On peut remarquer que le coefficient A de la plus haute dérivée de la fonction arbitraire est toujours égal à $1 + s s_0$. Il ne serait pas difficile d'ailleurs de former l'expression générale des coefficients $A, B, C, \ldots . L$, mais la formule générale ainsi obtenue paraît compliquée.

Supposons, par exemple, que dans la dernière des formules précédentes on prenne $f = f_0 = 1$; on aura pour u, en négligeant un facteur constant,

$$u = \frac{s^2 + s_0^2}{1 + s s_0}.$$

Les équations différentielles des lignes de courbure et des lignes asymptotiques de la surface obtenue seront respectivement

$$\begin{gathered} (1 + s_0^4)\,ds^2 = (1 + s^4)\,ds_0^2, \\ (1 + s_0^4)\,ds^2 + (1 + s^4)\,ds_0^2 - 4\,(s^2 + s_0^2)\,ds\,ds_0 = 0; \end{gathered}$$

on voit que la recherche des lignes de courbure se ramène à l'intégration de l'équation d'Euler. Ces lignes seront par conséquent algébriques.

7. Dans le Mémoire déjà cité plusieurs fois, M. Appell a rattaché d'une façon très remarquable les surfaces Σ_0 ou Σ_1 aux surfaces minima. Etant donnée en général une de nos surfaces Σ, correspondant à une valeur quelconque de m, nous allons voir qu'on peut en déduire deux surfaces contigües par une construction géométrique, qui comprend comme cas particulier la construction donnée par M. Appell. Voici comment on est conduit à ce résultat; nous avons vu qu'en désignant par V_m l'intégrale générale de l'équation $E(m)$, l'intégrale générale de l'équation $E(m+1)$ était donnée par la formule

$$V_{m+1} = \frac{\partial^2 V_m}{\partial\alpha\partial\beta},$$

sauf le cas où m était nul; mais si l'on écrit la relation précédente

$$V_{m+1} = \frac{\dfrac{\partial V_m}{\partial\alpha} - \dfrac{\partial V_m}{\partial\beta}}{\alpha - \beta}, \tag{26}$$

on reconnaît immédiatement qu'elle s'applique encore lorsque $m = 0$. On peut remarquer que les formules (18) et (26) permettent de ramener l'intégration de l'équation générale $E(m)$ au cas où m est compris entre 0 et 1.

Cela posé, soit v une intégrale quelconque de l'équation (12) et u l'intégrale de l'équation (9) qui lui correspond par le changement de variables défini par les formules (13). Au moyen de la formule (26) on déduira de v une intégrale v_1 de l'équation contigüe, puis une fonction u_1 qui vérifiera une nouvelle équation analogue à l'équation (9). En transformant convenablement la relation qui permet de déduire u_1 de u par le procédé qui nous a déjà servi plusieurs fois, on arrive à définir une construction géométrique analogue à celle de M. Appell. Mais on peut supprimer ces intermédiaires et partir directement de l'équation (9).

De l'équation (9)

$$(1 + ss_0)\frac{\partial^2 u}{\partial s\partial s_0} - s\frac{\partial u}{\partial s} - s_0\frac{\partial u}{\partial s_0} + \left[1 + \frac{2K}{1+ss_0}\right]u = 0 \tag{9}$$

on tire, en différentiant le premier membre par rapport à l'une quelconque des variables,

$$\left.\begin{aligned}(1 + ss_0)\frac{\partial^3 u}{\partial s\partial^2 s_0} &= s\frac{\partial^2 u}{\partial s^2} + \frac{2K}{1+ss_0}\left[\frac{s_0 u}{1+ss_0} - \frac{\partial u}{\partial s}\right],\\ (1 + ss_0)\frac{\partial^3 u}{\partial s\partial s_0^2} &= s_0\frac{\partial^2 u}{\partial s_0^2} + \frac{2K}{1+ss_0}\left[\frac{su}{1+ss_0} - \frac{\partial u}{\partial s_0}\right].\end{aligned}\right\} \tag{27}$$

Posons

$$U = u - s\frac{\partial u}{\partial s} - s_0\frac{\partial u}{\partial s_0} + \lambda\frac{ss_0 - 1}{1+ss_0}u, \tag{28}$$

λ désignant une constante quelconque. A l'aide des relations (9) et (27) on vérifie sans difficulté que la fonction U satisfait à l'équation suivante

$$\left.\begin{aligned}(1+ss_0)\frac{\partial^2 U}{\partial s\partial s_0}-s\frac{\partial U}{\partial s}-s_0\frac{\partial U}{\partial s_0}+\frac{ss_0+1+2(K+\lambda-1)}{1+ss_0}U\\=\frac{2(ss_0-1)}{(1+ss_0)^2}[\lambda^2-3\lambda-2K]u;\end{aligned}\right\}\tag{29}$$

si on prend pour λ une racine de l'équation (11) déjà obtenue

$$\lambda^2-3\lambda-2K=0,$$

on voit que la fonction U satisfait à une équation de même forme que l'équation (9)

$$(1+ss_0)\frac{\partial^2 U}{\partial s\partial s_0}-s\frac{\partial U}{\partial s}-s_0\frac{\partial U}{\partial s_0}+\frac{ss_0+1+2(K+\lambda-1)}{1+ss_0}U=0,\tag{30}$$

qui se déduit de l'équation (9) en remplaçant K par $K+\lambda-1$.

Soient λ', λ'' les deux racines de l'équation (11), que je suppose réelles et distinctes, et m', m'' les valeurs correspondantes de m,

$$m'=\lambda'-1,\quad m''=\lambda''-1,\quad m'+m''=1;$$

changer K en $K+\lambda'-1$, cela revient, d'après la relation (14), à remplacer m' par $m'+1$, et de même changer K en $K+\lambda''-1$ revient à remplacer m'' par $m''+1$. Par conséquent, si la fonction u d'où l'on est parti définit une surface Σ_m, les deux fonctions U que l'on vient d'obtenir définiront respectivement une surface Σ_{m+1} et une surface Σ_{m-1}. Pour avoir la signification géométrique de la formule (28) écrivons la comme il suit:

$$\frac{U}{1+ss_0}=\frac{u-s\dfrac{\partial u}{\partial s}-s_0\dfrac{\partial u}{\partial s_0}}{1+ss_0}+\lambda\frac{ss_0-1}{1+ss_0}\cdot\frac{u}{1+ss_0},$$

λ ayant une des deux valeurs $m+1$, $2-m$, et reportons-nous aux formules (3), (4), (5). L'expression

$$\frac{s\dfrac{\partial u}{\partial s}+s_0\dfrac{\partial u}{\partial s_0}-u}{1+ss_0}$$

représente la coordonnée z du point de la surface où le plan tangent coïncide avec le plan (4), $\frac{ss_0-1}{1+ss_0}$ est égal au cosinus de l'angle que fait la normale en ce point avec l'axe Oz ; enfin

$$\frac{u}{1+ss_0},\quad \frac{U}{1+ss_0}$$

sont les distances de l'origine aux deux plans considérés. On a donc la construction suivante : *Soit Σ_m une quelconque de nos surfaces, P un plan tangent à cette surface et M le point de contact. Menons un plan parallèle P_1 à une distance de l'origine égale à la distance du point M au plan xOy, diminuée de la projection sur l'axe Oz de la distance de l'origine au plan P multipliée par le facteur constant* $\frac{3}{2} \pm \left(m-\frac{1}{2}\right)$, *ce plan P_1 enveloppera une surface Σ_{m+1}, ou une surface Σ_{m-1}.*

Si on suppose $m=2$, ou $m=-1$, une des valeurs de λ sera nulle, la construction se simplifie, et on retrouve la construction de M. Appell pour passer d'une surface minima à une surface Σ_0 ou Σ_1. C'est d'ailleurs le seul cas où la construction se simplifie.

Réciproquement, on obtient *toutes* les surfaces Σ_{m+1}, ainsi que *toutes* les surfaces Σ_{m-1}, en appliquant la construction qui précède à toutes les surfaces Σ_m. Il suffit évidemment de le démontrer pour les surfaces Σ_{m+1}, par exemple. Soit U une intégrale de l'équation (30) ; il nous faut examiner si on peut trouver une fonction u vérifiant à la fois les équations (9) et (28). Ces équations peuvent s'écrire

$$\left.\begin{aligned} s\frac{\partial u}{\partial s}+s_0\frac{\partial u}{\partial s_0}&=u+\lambda\frac{ss_0-1}{1+ss_0}u-U,\\ (1+ss_0)\frac{\partial^2 u}{\partial s\,\partial s_0}&=\frac{\lambda(ss_0-1)-2K}{1+ss_0}u-U.\end{aligned}\right\}\tag{31}$$

De l'équation (28) on tire en différentiant

$$\frac{\partial U}{\partial s}=-s\frac{\partial^2 u}{\partial s^2}-s_0\frac{\partial^2 u}{\partial s\,\partial s_0}+\lambda\frac{ss_0-1}{1+ss_0}\frac{\partial u}{\partial s}+\frac{2\lambda s_0 u}{(1+ss_0)^2},$$

et on déduit de ces équations les valeurs de $\frac{\partial u}{\partial s}$, $\frac{\partial^2 u}{\partial s^2}$, $\frac{\partial^2 u}{\partial s\,\partial s_0}$ en fonction de u, U, s, s_0, $\frac{\partial u}{\partial s}$.

$$\begin{aligned} \frac{\partial u}{\partial s_0}&=\phi\left(u,\ U,\ s,\ s_0,\ \frac{\partial u}{\partial s}\right),\\ \frac{\partial^2 u}{\partial s\,\partial s_0}&=\psi\left(u,\ U,\ s,\ s_0,\ \frac{\partial u}{\partial s}\right),\\ \frac{\partial^2 u}{\partial s^2}&=\pi\left(u,\ U,\ s,\ s_0,\ \frac{\partial u}{\partial s}\right),\end{aligned}$$

et, en posant $\frac{\partial u}{\partial s} = \zeta$, u et ζ seront déterminées par un système d'équations aux différentielles totales

$$\begin{cases} du = \zeta ds + \phi(u, U, s, s_0, \zeta)\, ds_0, \\ d\zeta = \pi(u, U, s, s_0, \zeta)\, ds + \psi(u, U, s, s_0, \zeta)\, ds_0, \end{cases}$$

et, la fonction U étant supposée vérifier l'équation (30), les conditions d'intégrabilité seront satisfaites.

8. Si on applique aux surfaces Σ_{m+1} les deux constructions précédentes, en remplaçant m par $m+1$, on obtiendra les surfaces Σ_m et les surfaces Σ_{m+2}. On peut donc déduire les surfaces Σ_m des surfaces Σ_{m+1}, comme on déduit les surfaces Σ_{m+1} des surfaces Σ_m. Mais il est à remarquer qu'il n'y a pas réciprocité entre ces surfaces prises individuellement. Etant donnée une surface particulière Σ_m, la construction précédente appliquée à cette surface donne une surface bien déterminée Σ_{m+1}, et une nouvelle construction appliquée à Σ_{m+1} donnera une nouvelle surface Σ'_m qui sera en général différente de Σ_m. On le vérifiera plus loin sur des exemples.

On voit maintenant comment on pourra faire dériver, par des constructions géométriques successives, toutes nos surfaces Σ_m des surfaces pour lesquelles l'indice m est compris entre zéro et l'unité. On pourra même, en remarquant que les surfaces Σ_m et Σ_{1-m} sont identiques, diminuer cet intervalle de moitié.

Considérons en particulier les surfaces dont l'indice est un nombre entier. Attribuons à la constante m toutes les valeurs entières, tant négatives que positives; on a vu que chaque série de surfaces se présentait deux fois, pour les valeurs $1-m$ et m de l'indice. On peut alors se borner à considérer les constructions qui permettent de passer d'une série de surfaces à la série suivante. Chaque série de surfaces se déduira de la précédente par une construction bien déterminée et on peut faire dériver toutes ces surfaces d'une seule série, par exemple des surfaces minima.

Prenons en particulier les surfaces Σ_0 ou Σ_1; la valeur générale de u est, comme on l'a vu plus haut,

$$u = (1 + ss_0)[f(s) + f_0(s_0)];$$

l'équation (11) est ici

$$\lambda^2 - 3\lambda + 2 = 0.$$

Si on prend $\lambda = 1$, on trouve

$$U = -(1 + ss_0)[sf'(s) + s_0 f_0'(s_0)],$$

qui convient encore aux surfaces Σ_0; on s'explique aisément ce fait en remarquant que ces surfaces sont contigües à elles-mêmes. Si on prend ensuite $\lambda = 2$, on trouve

$$U = 2s_0 s f(s) + 2ss_0 f_0(s_0) - (1 + ss_0)[f(s) + sf'(s) + s_0 f_0'(s_0) + f_0(s_0)];$$

c'est, avec un changement de notations, la valeur générale de u qui convient aux surfaces minima. Prenons encore les surfaces minima; la valeur générale de u est de la forme

$$u = 2sf_0(s_0) + 2s_0 f(s) - (1 + ss_0)[f'(s) + f_0'(s_0)].$$

L'équation en λ est ici $\lambda^2 - 3\lambda = 0$. Pour $\lambda = 0$, on a

$$U = (1 + ss_0)[sf''(s) - f'(s) + s_0 f_0''(s_0) - f_0'(s_0)];$$

c'est la forme qui convient aux surfaces Σ_0. Pour $\lambda = 3$, on trouve

$$U = (1 + ss_0)[sf''(s) + 2f'(s) + s_0 f_0''(s_0) + 2f_0'(s_0)] - 6s_0[sf'(s) + f(s)]$$
$$- 6s[s_0 f_0'(s_0) + f_0(s_0)] + \frac{12ss_0}{1 + ss_0}[s_0 f(s) + sf_0(s_0)].$$

Cette valeur de U peut s'écrire, en posant $sf(s) = \phi(s)$, $s_0 f_0(s_0) = \phi_0(s_0)$,

$$U = (1 + ss_0)[\phi''(s) + \phi_0''(s_0)] - 6[s\phi_0'(s_0) + s_0\phi'(s)] + 12\frac{s^2\Phi_0(s_0) + s_0^2\Phi(s)}{1 + ss_0};$$

c'est précisément la valeur générale de u trouvée plus haut qui convient aux surfaces Σ_3.

9. Etant données une courbe gauche analytique C et une développable Δ passant par cette courbe, il existe en général une surface minima et une seule tangente à la développable Δ le long de la courbe C; les coordonnées d'un point de cette surface peuvent être exprimées en fonction de deux paramètres par des formules ne renfermant que des quadratures. Cette importante question a été, comme on sait, posée et résolue pour la première fois par E. G. Bjorling.* M. Appell a résolu le même problème pour les surfaces Σ_0.† Au moyen d'un très-beau résultat dû à Riemann, on peut traiter la même question pour une surface quelconque satisfaisant à l'équation (9), quelle que soit la valeur de la constante K.

Imaginons que par l'origine on mène des perpendiculaires à tous les plans tangents de la développable Δ; ces droites forment un cône, qui coupe la sphère S de rayon égal à l'unité suivant une certaine courbe c, qui sera définie par une relation de la forme

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* *Archives de Grunert*, t. IV, p. 290; 1844.

† *Mémoire sur les déblais et les remblais.* Voyez *Mémoires présentés par divers savants à l'Académie des Sciences*, t. XXIX, No. 3, p. 137.

$$\phi(s, s_0) = 0;$$

la développable Δ étant donnée, la fonction cherchée u prendra des valeurs connues pour les systèmes de valeurs de s et de s_0 qui vérifient la relation précédente. D'un autre côté, la courbe C étant donnée, on connaîtra aussi la valeur de z pour ces systèmes de valeurs de s et de s_0, c'est-à-dire qu'on connaîtra, le long de la courbe c, la fonction

$$s\frac{\partial u}{\partial s} + s_0\frac{\partial u}{\partial s_0} - u.$$

Cette relation, jointe à la relation

$$du = \frac{\partial u}{\partial s}ds + \frac{\partial u}{\partial s_0}ds_0,$$

nous fera connaître les valeurs des dérivées partielles de la fonction inconnue $\frac{\partial u}{\partial s}$, $\frac{\partial u}{\partial s_0}$, le long de c, à moins que l'on ait

$$sds_0 - s_0ds = 0.$$

Je laisse de côté ce cas singulier, qui ne se présentera pas si la courbe C et la développable Δ sont réelles. Le problème de Géométrie posé plus haut est donc ramené au problème d'Analyse suivant:

Déterminer une fonction u *satisfaisant à l'equation* (9) *et prenant, ainsi que ses dérivées premières* $\frac{\partial u}{\partial s}$, $\frac{\partial u}{\partial s_0}$, *des valeurs données à l'avance le long d'une courbe* c *représentée par l'équation*

$$\phi(s, s_0) = 0.$$

Il suffit d'ailleurs de se donner l'une des dérivées $\frac{\partial u}{\partial s}$, $\frac{\partial u}{\partial s_0}$, car la relation écrite plus haut

$$du = \frac{\partial u}{\partial s}ds + \frac{\partial u}{\partial s_0}ds_0,$$

appliquée à un déplacement le long de cette courbe, fera connaître celle des deux dérivées partielles qui n'est pas donnée *a priori*. Au moyen des formules (13), ce problème se ramène lui-même au problème analogue relatif à l'équation (12), car la relation $\phi(s, s_0) = 0$ se change en une certaine relation

$$\psi(\alpha, \beta) = 0;$$

et, si la fonction u et ses dérivées partielles du premier ordre par rapport à s

et à s_0 sont supposées connues pour tous les systèmes de valeurs de s et de s_0 vérifiant l'équation $\phi(s, s_0) = 0$, il en sera évidemment de même de la fonction v et de ses dérivées partielles prises par rapport à α et à β pour tous les systèmes de valeurs de α et de β vérifiant la relation nouvelle $\psi(\alpha, \beta) = 0$.

Ce dernier problème se trouve résolu, sous une forme un peu différente, dans le Mémoire déjà cité de Riemann (*Gesammelte Werke*, p. 145). Le grand géomètre établit que l'on peut obtenir la fonction v, satisfaisant aux conditions précédentes, par des quadratures seulement.

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Remarks on the Logarithmic Integrals of Regular Linear Differential Equations.

BY KARL HEUN, *Munich.*

Logarithms generally appear in the expressions for the solutions of regular linear differential equations when two or more roots of the fundamental equation become equal to each other. But if the corresponding indices differ by integers (not including zero), these logarithms may disappear, provided that certain conditions be satisfied. Fuchs has expressed these conditions in the form of determinants (Journal für Mathematik, LXVIII, pag. 376). I noticed, however, some time ago, that the Fuchs equations are not independent of each other when the proposed differential equation is of a higher order than the second. In the present paper the minimum number of conditions on which the existence of logarithms depends is deduced from very elementary principles. Besides this, several theorems concerning the pseudo-singular points (points à apparence singulière) of linear differential equations of the second order are given in such an explicit form as to facilitate practical applications to concrete cases.

1.

Any regular linear differential equation may be reduced to the form (cf. my paper "Zur Theorie der mehrwerthigen, mehrfach lineär verknüpften Functionen" in Acta Mathematica, t. XI, pag. 97),

$$[\psi(x)]^p \cdot \frac{d^p y}{dx^p} + [\psi(x)]^{p-1} \cdot F_1(x) \cdot \frac{d^{p-1} y}{dx^{p-1}} + \ldots\ldots + F_p(x) \cdot y = 0, \qquad \text{(A)}$$

$F_\pi(x)$ denoting an entire function of the degree $\pi(i-1)$. The i roots of the equation

$$\psi(x) = (x - \xi_1)(x - \xi_2) \ldots\ldots (x - \xi_i) = 0,$$

together with $\xi_{i+1} = \infty$ mark the "singular points" of the integral function y which satisfies the preceding differential equation.

Let $(y_{1\mathrm{i}}, y_{2\mathrm{i}}, \ldots . y_{p\mathrm{i}})$ be a system of fundamental integrals of equation (A), representing its complete solution in the neighborhood of the singular point $\xi_\mathrm{i}\,(\mathrm{i}=1,\ 2, \ldots . i,\ i+1)$, and suppose the point P, the geometrical representation of the argument x, to describe round the singular point ξ_i, a closed curve whose interior does not contain any other singular point $\xi_\iota\ (\iota \gtrless \mathrm{i})$, then the integrals $y_{1\mathrm{i}}, y_{2\mathrm{i}}, \ldots . y_{p\mathrm{i}}$ will not return to their initial values when the point P has completed its circuit, but will acquire the new values

$$\bar{y}_{1\mathrm{i}} = w_\mathrm{i}^{(1)} . y_{1\mathrm{i}},\ \bar{y}_{2\mathrm{i}} = w_\mathrm{i}^{(2)} . y_{2\mathrm{i}},\ \ldots .\ \bar{y}_{p\mathrm{i}} = w_\mathrm{i}^{(p)} . y_{p\mathrm{i}}. \tag{a}$$

The coefficients $w_\mathrm{i}^{(1)}, w_\mathrm{i}^{(2)}, \ldots . w_\mathrm{i}^{(p)}$ are the roots of a certain algebraic equation which is generally termed the "fundamental equation."

Whenever two or more (π) roots of this equation become equal to each other, the equations (a) will have to be replaced by the following system (cf. Fuchs, Journal für Mathemat., t. LXVI, pag. 136]:

$$\left.\begin{array}{ll}
\bar{y}_{1\mathrm{i}} & = w_\mathrm{i} \cdot y_{1\mathrm{i}}, \\
\bar{y}_{2\mathrm{i}} & = w_\mathrm{i} \cdot y_{2\mathrm{i}} + \tilde{\omega}_{11} y_{1\mathrm{i}}, \\
\bar{y}_{3\mathrm{i}} & = w_\mathrm{i} \cdot y_{3\mathrm{i}} + \tilde{\omega}_{21} y_{2\mathrm{i}} + \tilde{\omega}_{12} y_{1\mathrm{i}}, \\
\multicolumn{2}{l}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\
\bar{y}_{\pi,\mathrm{i}} & = w_\mathrm{i} y_{\pi\mathrm{i}} + \tilde{\omega}_{\pi-1,1} \cdot y_{\pi-1,\mathrm{i}} + \tilde{\omega}_{\pi-2,2} \cdot y_{\pi-2,\mathrm{i}} + \ldots + \tilde{\omega}_{11\pi-1} \cdot y_{1\mathrm{i}}, \\
\bar{y}_{\pi+1,\mathrm{i}} & = w_\mathrm{i}^{(\pi+1)} . y_{\pi+1,\mathrm{i}}, \\
\multicolumn{2}{l}{\cdots\cdots\cdots\cdots\cdots} \\
\bar{y}_{p,\mathrm{i}} & = w_\mathrm{i}^{(p)} . y_{p,\mathrm{i}}.
\end{array}\right\} \quad \text{(I)}$$

In these formulas $\tilde{\omega}_{11}, \tilde{\omega}_{21}, \tilde{\omega}_{12}, \ldots . \tilde{\omega}_{1,\pi-1}$ are constant quantities and

$$w_\mathrm{i} = w_\mathrm{i}^{(1)} = w_\mathrm{i}^{(2)} = \ldots . = w_\mathrm{i}^{(\pi)}. \tag{b}$$

The equations (I) will be satisfied by the following expressions for $y_{1\mathrm{i}}, y_{2\mathrm{i}}, \ldots . y_{p\mathrm{i}}$:

$$\left.\begin{array}{ll}
y_{1\mathrm{i}} & = \eta_{1\mathrm{i}}, \\
y_{2\mathrm{i}} & = \eta_{2\mathrm{i}} + c_{11} \cdot \eta_{1\mathrm{i}} \cdot \lg(x - \xi_\mathrm{i}), \\
y_{3\mathrm{i}} & = \eta_{3\mathrm{i}} + c_{21} \cdot \eta_{2\mathrm{i}} \cdot \lg(x - \xi_\mathrm{i}) + c_{12} \cdot \eta_{1\mathrm{i}} \cdot [\lg(x - \xi_\mathrm{i})]^2, \\
\multicolumn{2}{l}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\
y_{\pi,\mathrm{i}} & = \eta_{\pi,\mathrm{i}} + c_{\pi-1,1} \cdot \eta_{\pi-1,\mathrm{i}} \cdot \lg(x - \xi_\mathrm{i}) + \ldots + c_{11\pi-1} \cdot \eta_{1\mathrm{i}} \cdot [\lg(x - \xi_\mathrm{i})]^{\pi-1}, \\
y_{\pi+1,\mathrm{i}} & = \eta_{\pi+1,\mathrm{i}}, \\
\multicolumn{2}{l}{\cdots\cdots\cdots} \\
y_{p,\mathrm{i}} & = \eta_{p,\mathrm{i}}.
\end{array}\right\} \quad \text{(II)}$$

The function $\eta_{\mathfrak{p}\mathrm{i}}\,(\mathfrak{p}=1,\,2,\,\ldots.\,p)$ is defined by the expression

$$\eta_{\mathfrak{p}\mathrm{i}}=(x-\xi_\mathrm{i})^{\lambda_{\mathfrak{p}\mathrm{i}}}.\,\mathfrak{P}_\mathfrak{p}\,(x-\xi_\mathrm{i}), \tag{c}$$

$\mathfrak{P}_\mathfrak{p}\,(x-\xi_\mathrm{i})$ being the usual symbol for the convergent series

$$a_0^{(\mathfrak{p})}+a_1^{(\mathfrak{p})}\,(x-\xi_\mathrm{i})+a_2^{(\mathfrak{p})}\,(x-\xi_\mathrm{i})^2+\ldots.+\text{ in inf.},$$

where $a_0^{(\mathfrak{p})}$ is always different from zero.

The indices $\lambda_{\mathfrak{p}\mathrm{i}}\,(\mathfrak{p}=1,\,2,\,\ldots.\,p)$ are the roots of the equation

$$\left.\begin{aligned}&[\psi'(\xi_\mathrm{i})]^p\lambda\,(\lambda-1)\ldots.(\lambda-p+1)\\&+\sum_{\mathfrak{p}=1}^{\mathfrak{p}=p-1}[\psi'\,(\xi_\mathrm{i})]^{p-\mathfrak{p}}.F_\pi\,(\xi_\mathrm{i})\,.\,\lambda\,(\lambda-1)\ldots.(\lambda-p+\pi+1)+F_p(\xi_\mathrm{i})=0.\end{aligned}\right\} \tag{B}$$

Because $\lambda_{\mathfrak{p}\mathrm{i}}$ and $w_\mathrm{i}^{(\mathfrak{p})}$ are connected by the relation

$$\lambda_{\mathfrak{p}\mathrm{i}}=\frac{1}{2\pi\sqrt{-1}}\,.\,\lg w_\mathrm{i}^{(\mathfrak{p})}, \tag{d}$$

the indices $\lambda_{1,\,\mathrm{i}},\lambda_{2,\,\mathrm{i}},\ldots.\,\lambda_{\pi,\,\mathrm{i}}$ can only differ from one another by integers.

The constant coefficients $c_{11},\,c_{21},\,c_{12},\,\ldots.\,c_{1,\,\pi-1}$ in equations (II) are subjected to certain conditions, unless $\pi=2$. In fact we derive from the system (II)

$$\bar{y}_{3\mathrm{i}}=\bar{\eta}_{3\mathrm{i}}+c_{21}\bar{\eta}_{2\mathrm{i}}\,.\,[\lg\,(x-\xi_\mathrm{i})+2\pi\sqrt{-1}]+c_{12}\bar{\eta}_{1\mathrm{i}}\,[\lg\,(x-\xi_\mathrm{i})+2\pi\sqrt{-1}]^2.$$

But from equation (c) we conclude

$$\bar{\eta}_{\mathfrak{p}\mathrm{i}}=w_i\,.\,\eta_{\mathfrak{p}\mathrm{i}}\,|\,\mathfrak{p}=1,\,2,\,\ldots.\,\pi.$$

Hence

$$\begin{aligned}\bar{y}_{3\mathrm{i}}&=w_\mathrm{i}\eta_{3\mathrm{i}}+c_{21}w_\mathrm{i}\,.\,\eta_{2\mathrm{i}}\,[\lg\,(x-\xi_\mathrm{i})+2\pi\sqrt{-1}]\\&\qquad+c_{12}w_\mathrm{i}\,.\,\eta_{1\mathrm{i}}\,[\lg\,(x-\xi_\mathrm{i})+2\pi\sqrt{-1}]^2\\&=w_\mathrm{i}\eta_{3\mathrm{i}}+2\pi\sqrt{-1}\,w_\mathrm{i}\{c_{21}\eta_{2\mathrm{i}}+2c_{12}\eta_{1\mathrm{i}}\,.\lg\,(x-\xi_\mathrm{i})\}\\&\qquad+(2\pi\sqrt{-1})^2w_i\,.\,\eta_{1i}\,[\lg\,(x-\xi_i)]^2.\end{aligned}$$

The last expression for $\bar{y}_{3\mathrm{i}}$ will only coincide with

$$w_\mathrm{i}\,.\,\eta_{3\mathrm{i}}+\tilde{\omega}_{21}\eta_{2\mathrm{i}}+\tilde{\omega}_{12}\eta_{1i}$$

if the condition $2c_{12}=c_{21}\,.\,c_{11}$ be fulfilled. By continuing this simple process we get the remaining relations

$$2c_{22}=c_{31}\,.\,c_{21},\quad 3c_{13}=c_{22}\,.\,c_{11},\quad\ldots.\quad\pi c_{1,\,\pi-1}=c_{2,\,\pi-2}\,.\,c_{11}.$$

The number of the coefficients $c_{11},\,c_{21},\,c_{12},\,\ldots.\,c_{1,\,\pi-1}$ is $\frac{1}{2}\,\pi\,(\pi-1)$. As there are $\frac{1}{2}(\pi-1)(\pi-2)$ conditions between them, only $\pi-1$ of them are independent of one another.

Now it is evident that y_{2i} will contain no logarithm if $c_{11}=0$. Likewise y_{3i} will be free of logarithms if $c_{11}=c_{21}=0$. Generally the integral of the form

$$y_{\pi i}=\eta_{\pi i}+c_{\pi-1,1}\cdot\eta_{\pi-1,i}\cdot\lg(x-\xi_i)+\dots+c_{1,\pi-1}\cdot\eta_{1i}\cdot[\lg(x-\xi_i)]^{\pi-1}$$

can be reduced to the form $y_{\pi i}=\eta_{\pi i}$ if the preceding integrals $y_{\pi-1,i},\dots y_2$ contain no logarithms and if the additional condition $c_{\pi-1,1}=0$ be fulfilled.

By combining this result with a well known theorem of Fuchs (Journal für Mathematik, t. LXVIII, pag. 367), we obtain the following theorem:

I. "If the indices $\lambda_{1i}, \lambda_{2i},\dots\lambda_{\pi i}$ belonging to the functions $y_{1i}, y_{2i},\dots y_{\pi}$ differ from one another by integer numbers, the analytical expressions of those integrals will, as a rule, contain logarithms (y_{1i} excepted). But unless two of the indices $\lambda_{1i}, \lambda_{2i},\dots\lambda_{\pi i}$ are equal to one another, they may be freed from the logarithms by satisfying certain conditions. The number of these conditions is $\pi-1$."

We have not yet proved that the expressions given in equations (II) for $y_{1i}, y_{2i},\dots y_{pi}$ form really a complete system of fundamental integrals of the differential equation (A). For that purpose, let us consider the following expressions:

$$y_{1i}\cdot(x-\xi_i)^{-\lambda_{1i}},\ y_{2i}\cdot(x-\xi_i)^{-\lambda_{2i}},\ \dots\ y_{\pi i}\cdot(x-\xi_i)^{-\lambda_{\pi i}}.$$

If $\lambda_{1i}>\lambda_{2i}>\dots>\lambda_{\pi i}$, we may write

$$\lambda_{\pi-1,i}-\lambda_{\pi i}=n_1,\ \lambda_{\pi-2,i}-\lambda_{\pi i}=n_2,\ \dots\ \lambda_{1i}-\lambda_{\pi i}=n_{\pi-1},$$

where $n_1, n_2,\dots n_{\pi-1}$ are certain positive integers subject to the conditions

$$n_1<w_2,\ n_2<n_3,\ \dots\ n_{\pi-2}<n_{\pi-1},$$

or

$$n_1\geqq 1,\quad n_2\geqq 2,\ \dots\ n_{\pi-1}\geqq\pi-1.$$

For any index $\tilde{\omega}\leqq\pi$ we have by equation (c),

$$\begin{aligned}y_{\tilde{\omega}i}=(x-\xi_i)^{\lambda_{\tilde{\omega}i}}\{\mathfrak{P}_{\tilde{\omega}}(x-\xi_i)&+c_{\tilde{\omega}-1,1}\cdot(x-\xi_i)^{n_1}\cdot\mathfrak{P}_{\tilde{\omega}-1}(x-\xi_i)\cdot\lg(x-\xi_i)\\&+c_{\tilde{\omega}-2,2}\cdot(x-\xi_i)^{n_2}\cdot\mathfrak{P}_{\tilde{\omega}-2}(x-\xi_i)\cdot[\lg(x-\xi_i)]^2\\&+\dots\dots\dots\dots\dots\\&+c_{1,\tilde{\omega}-1}\cdot(x-\xi_i)^{n_{\tilde{\omega}-1}}\cdot\mathfrak{P}_1(x-\xi_i)\cdot[\lg(x-\xi_i)]^{\tilde{\omega}-1}\}.\end{aligned}$$

Hence we conclude that the expression $y_{\tilde{\omega}i}\cdot(x-\xi_i)^{-\lambda_{\tilde{\omega}i}}=\Omega_{\tilde{\omega}}$ will be a finite continuous function in the neighborhood of the singular point ξ_i, for we have under the preceding conditions,

$$[(x-\xi_i)^{n_y}.[\lg(x-\xi_i)]^{y-1}]_{x=\xi_i}=0,$$
$$(y=1,\ 2,\ \ldots.\ \tilde{\omega}).$$

From the equation

$$y_{\tilde{\omega}i}=(x-\xi_i)^{\lambda_{\tilde{\omega}i}}.\Omega_{\tilde{\omega}}$$

we derive by differentiation

$$\frac{dy_{\tilde{\omega}i}}{dx}=(x-\xi_i)^{\lambda_{\tilde{\omega}i}-1}.\left\{\lambda_{\tilde{\omega}i}.\Omega_{\tilde{\omega}}+(x-\xi_i).\frac{d\Omega_{\tilde{\omega}}}{dx}\right\}.$$

It will be easily seen that the expression enclosed in brackets has the same characteristic properties as the function $\Omega_{\tilde{\omega}}$ itself. On this account we may write

$$\frac{dy_{\tilde{\omega}i}}{dx}=(x-\xi_i)^{\lambda_{\tilde{\omega}i}-1}.\Omega_{\tilde{\omega}}^{(\prime)},$$

and likewise

$$\frac{d^2y_{\tilde{\omega}i}}{dx^2}=(x-\xi_i)^{\lambda_{\tilde{\omega}i}-2}.\Omega_{\tilde{\omega}}^{(\prime\prime)},\ \ldots.\ \frac{d^py_{\tilde{\omega}i}}{dx^p}=(x-\xi_i).^{\lambda_{\tilde{\omega}i}-p}.\Omega_{\tilde{\omega}}^{(p)},$$

$\Omega_{\tilde{\omega}}^{(\prime)}.\Omega_{\tilde{\omega}}^{(\prime\prime)},\ \ldots.\ \Omega_{\tilde{\omega}}^{(p)}$ denoting a series of functions analogous to $\Omega_{\tilde{\omega}}$. Hence we prove in the usual manner that $y_{1i},\ y_{2i},\ \ldots.\ y_{\pi i}$, together with $y_{\pi+1,i}$, $y_{\pi+2,i},\ \ldots.\ y_{pi}$, satisfy a regular linear differential equation of the order p (cf. Fuchs' paper in Vol. 66 of the Journal für Mathematik, pag. 142–144).

The case $\pi=p$ requires our special attention. If now $y_{2i},\ y_{3i},\ \ldots.\ y_{pi}$ do not involve any logarithms, the point ξ_i will cease to be a singular point for the ratios $y_{1i}:y_{pi},\ y_{2i}:y_{pi},\ \ldots.\ y_{p-1,i}:y_{pi}$. It may be called a "pseudo-singular point" of the differential equation (A). Remembering that the indices λ_{1i}, $\lambda_{2i},\ \ldots.\ \lambda_{pi}$ are now such as to satisfy the $p-1$ conditions

$$\lambda_{p-1,i}-\lambda_{pi}=n_1,\ \lambda_{p-2,i}-\lambda_{pi}=n_2,\ \ldots.\ \lambda_{1i}-\lambda_{pi}=n_{p-1}; \qquad \text{(C)}$$

$n_1,\ n_2,\ \ldots.\ n_{p-1}$ being positive integers, we conclude from theorem I:

II. "In order that the point ξ_i be a *pseudo-singular point* of the linear equation (A), it is necessary and sufficient to satisfy $2p-2$ conditions."

We shall have now to establish $p-1$ of these conditions "explicitly" so as to acquire a sufficient criterion, whether a certain point ξ_i be a pseudo-singular point of a regular differential equation or not, supposing that the $p-1$ conditions (C) are fulfilled *a priori*. This problem will be solved gradually by beginning with equations of the second order.

2.

In the particular case $p = 2$ equation (A) takes the simple form

$$[\psi(x)]^2 \cdot \frac{d^2y}{dx^2} + \psi(x) \cdot F_1(x) \cdot \frac{dy}{dx} + F_2(x) \cdot y = 0. \tag{1}$$

Equation (B) becomes

$$[\psi'(\xi_{\mathrm{i}})]^2 \lambda(\lambda - 1) + \psi'(\xi_{\mathrm{i}}) \cdot F_1(\xi_{\mathrm{i}}) \cdot \lambda + F_2(\xi_{\mathrm{i}}) = 0, \qquad \mathrm{i} = 1, 2, \ldots\ldots i. \tag{2}$$

The analogous equation belonging to the point $\xi_{i+1} = \infty$ is

$$\lambda(\lambda + 1) - \left[\frac{F_1(x)}{x^{i-1}}\right]_{x=\infty} \cdot \lambda + \left[\frac{F_2(x)}{x^{2i-2}}\right]_{x=\infty} = 0. \tag{3}$$

From this equation (whose roots are $\lambda_{1,i+1}$, $\lambda_{2,i+1}$) follows

$$\left[\frac{F_2(x)}{x^{2i-2}}\right]_{x=\infty} = \lambda_{1,i+1} \cdot \lambda_{2,i+1},$$

whilst we get from equation (2)

$$F_2(\xi_{\mathrm{i}}) = \lambda_{1,\mathrm{i}} \cdot \lambda_{2,\mathrm{i}} \,|\, \mathrm{i} = 1, 2, \ldots\ldots i.$$

The integer function $F_2(x)$ will therefore be divisible by $\psi(x)$ if

$$\lambda_{11} = \lambda_{12} = \ldots\ldots = \lambda_{1,i-1} = \lambda_{1i} = 0.$$

But it is very easy to express the general integral of equation (1) by another function whose indices satisfy the latter condition. Indeed, denoting every general solution of a regular linear differential equation by the tables of its indices in this manner

$$\begin{Bmatrix} \lambda_{11}, & \lambda_{12}, & \ldots\ldots & \lambda_{1i}, & \lambda_{1,i+1} \\ \lambda_{21}, & \lambda_{22}, & \ldots\ldots & \lambda_{2i}, & \lambda_{2,i+1} \end{Bmatrix}$$

we may verify immediately the following relation:

$$\begin{Bmatrix} \lambda_{11}; & \lambda_{12}, & \ldots\ldots & \lambda_{1i}, & \lambda_{1,i+1} \\ \lambda_{21}, & \lambda_{22}, & \ldots\ldots & \lambda_{2i}, & \lambda_{2,i+1} \end{Bmatrix} = (x - \xi_1)^{\lambda_{11}} \cdot (x - \xi_2)^{\lambda_{12}} \ldots\ldots (x - \xi_i)^{\lambda_{1i}}$$

$$\times \begin{Bmatrix} 0 & , \ldots\ldots 0 \,. & \lambda_{1,i+1} + \sum_{\mathrm{i}=1}^{\mathrm{i}=i} \lambda_{1\mathrm{i}} \\ \lambda_{21} - \lambda_{11}, & \ldots\ldots \lambda_{2i} - \lambda_{1i}, & \lambda_{2,i+1} + \sum_{\mathrm{i}=1}^{\mathrm{i}=i} \lambda_{1\mathrm{i}} \end{Bmatrix}$$

If we put therefore

$$y = (x-\xi_1)^{\lambda_{11}} . (x-\xi_2)^{\lambda_{12}} \dots (x-\xi_i)^{\lambda_{1i}} . \eta ,$$

the function η will satisfy the differential equation

$$\psi(x) . \frac{d^2\eta}{dx^2} + \chi(x) . \frac{d\eta}{dx} + \tilde{\omega}(x) . \eta = 0, \tag{1a}$$

$$\left[\chi(x) = F_1(x);\ \tilde{\omega}(x) = \frac{F_2(x)}{\psi(x)}\right],$$

$\tilde{\omega}(x)$ is an integer function of the degree $i-2$. On account of equation (3a) it has the form

$$\tilde{\omega}(x) = \mathfrak{l}_{1,\,i+1} . \mathfrak{l}_{2,\,i+1} . x^{i-2} + \varkappa_1 x^{i-3} + \varkappa_2 x^{i-4} + \dots + \varkappa_{i-3} x + \varkappa_{i-2} \tag{1b}$$

if the function η is denoted by

$$\left\{\begin{matrix} 0, & 0, & \dots & 0, & \mathfrak{l}_{1,\,i+1} \\ \mathfrak{l}_1, & \mathfrak{l}_2, & \dots & \mathfrak{l}_i, & \mathfrak{l}_{2,\,i+1} \end{matrix}\right\}.$$

The function $\chi(x)$ is determined by the equations

$$\chi(\xi_{\mathrm{i}}) = -\psi'(\xi_{\mathrm{i}})\, \mathfrak{l}_{\mathrm{i}} . \tag{2a}$$

The constant coefficients $\varkappa_1, \varkappa_2, \dots \varkappa_{i-2}$ are independent of the indices $\mathfrak{l}$. According to the special values which are attributed to them, the character of the function η will be more or less complicated. For this reason we shall term them the *characteristic parameters* of the differential equation (1a).

After this digression we return to the original differential equation (1). Let ξ_ι be any one of the points $\xi_1, \xi_2, \dots \xi_i$ and

$$y_{1\iota} = (x-\xi_\iota)^{\lambda_{1\iota}} . \mathfrak{P}_1(x-\xi_\iota) \tag{4}$$

be one of the fundamental integrals belonging to the point ξ_ι. Then the second integral $y_{2\iota}$ may be represented by the expression

$$y_{2\iota} = y_{1\iota} . \int \frac{e^{-\int \frac{F_1(x)}{\psi(x)} dx}}{y_{1\iota} . y_{1\iota}}\, dx . \tag{5}$$

Since the roots of the equation $\psi(x) = 0$ are different from one another, we have

$$\frac{F_1(x)}{\psi(x)} = \sum_{\mathrm{i}=1}^{\mathrm{i}=i} \frac{F_1(\xi_{\mathrm{i}})}{\psi'(\xi_{\mathrm{i}})} . \frac{1}{x-\xi_{\mathrm{i}}} . \tag{6}$$

But, on account of equation (2), we find directly

$$F_1(\xi_{\mathrm{i}}) = -\left[\lambda_{1\mathrm{i}} + \lambda_{2\mathrm{i}} - 1\right] . \psi'(\xi_{\mathrm{i}}),$$

consequently

$$e^{-\int \frac{F_1(x)}{\psi(x)}dx} = \prod_{\mathrm{i}=1}^{\mathrm{i}=i} (x-\xi_{\mathrm{i}})^{\lambda_{1\mathrm{i}}+\lambda_{2\mathrm{i}}-1}.$$

Writing for brevity

$$P(x) = \frac{\prod_{\mathrm{i}=1}^{\mathrm{i}=i} (x-\xi_{\mathrm{i}})^{\lambda_{1\mathrm{i}}+\lambda_{2\mathrm{i}}-1}}{(x-\xi_\iota)^{\lambda_{1\iota}+\lambda_{2\iota}-1}} \cdot \frac{1}{\mathfrak{P}_1(x-\xi_\iota)\,.\,\mathfrak{P}_1(x-\xi_\iota)}, \tag{7}$$

equation (5) can be put in the form

$$y_{2\iota} = y_{1\iota} \cdot \int P(x)\,.\,(x-\xi_\iota)^{\lambda_{1\iota}-\lambda_{1\iota}-1}\,dx.$$

But it is possible to develop $P(x)$ into the convergent series

$$c_0 + c_1(x-\xi_\iota) + c_2(x-\xi_\iota)^2 + \dots + \text{in inf.}$$

where the first constant c_0 differs necessarily from zero. Therefore we have

$$y_{2\iota} = y_{1\iota} \cdot \int \Big[\sum_{\nu=0}^{\nu=\infty} c_\nu\,.\,(x-\xi_\iota)^{\lambda_{2\iota}-\lambda_{1\iota}-1+\nu}\Big]\,dx.$$

Supposing $\lambda_{1\iota} - \lambda_{2\iota} = n$ (n being a positive integer) we obtain, by integration,

$$y_{2\iota} = y_{1\iota} \cdot \Big\{ -\frac{1}{n} c_0 (x-\xi_\iota)^{-n} - \frac{1}{n-1} c_1 (x-\xi_\iota)^{-(n-1)} - \dots - c_{n-1}(x-\xi_\iota)^{-1}$$
$$+ c_n\,.\,\lg(x-\xi_\iota) + c_{n+1}(x-\xi_\iota) + \frac{1}{2} c_{n+2}(x-\xi_\iota) + \dots + \text{in inf.}\Big\}.$$

Hence we get for $y_{2\iota}$ the final expression

$$y_{2\iota} = (x+\xi_\iota)^{\lambda_{2\iota}}\,.\,\mathfrak{P}_2(x-\xi_\iota) + c_n (x-\xi_\iota)^{\lambda_{1\iota}}\,.\,\mathfrak{P}_1(x-\xi_\iota)\,.\,\lg(x-\xi_\iota). \tag{8}$$

From this formula it is evident that $y_{2\iota}$ will contain no logarithm if the equation $c_n = 0$ be fulfilled. We have now to establish the connection of this equation with the coefficients of the differential equation (1) or, what is the same thing, with the indices λ and the characteristic parameters $\varkappa$.

We may always consider $\mathfrak{P}_1(x-\xi_\iota)$ as a known function. Let it be given in the form

$$\mathfrak{P}_1(x-\xi_\iota) = a_0 + a_1(x-\xi_\iota) + a_2(x-\xi_\iota)^2 + \dots + \text{in inf.}$$

$\mathfrak{P}_1(x-\xi_\iota)\,.\,\mathfrak{P}_1(x-\xi_\iota)$ will be likewise a convergent series of the form

$$s - \alpha_0 + \alpha_1(x-\xi_\iota) + \alpha_2(x-\xi_\iota)^2 + \dots + \text{in inf.}$$

We have $\alpha_0 = a_0 a_0$, $\alpha_1 = 2a_1$ and generally

$$\begin{aligned} \alpha_{2\nu} &= 2\left[a_0 a_{2\nu} + a_1 a_{2\nu-1} + \ldots + a_{\nu-1} a_{\nu+1}\right] + a_\nu a_\nu, \\ \alpha_{2\nu+1} &= 2\left[a_0 a_{2\nu+1} + a_1 a_{2\nu} + \ldots + a_\nu a_{\nu+1}\right]. \end{aligned}$$

The coefficients of the series

$$\frac{1}{s^2} = A_0 - A_1 (x - \xi_\iota) + A_2 (x - \xi_\iota)^2 - A_3 (x - \xi_\iota)^3 + \ldots \pm \text{ in inf.}$$

are connected with those of the series s by the formulas $[\alpha_0 = 1]$,

$$A_0 = 1;\ A_1 = \alpha_1;\ A_2 = \alpha_1^2 - \alpha_2;\ A_3 = \alpha_1^3 - 2\alpha_1\alpha_2 + \alpha_3, \text{ etc.}$$

The general expression for A_n as a function of $\alpha_1, \alpha_2, \ldots \alpha_n$ is

$$A_n = \begin{vmatrix} \alpha_1, & \alpha_2, & \alpha_3, & \ldots & \alpha_n \\ 1, & \alpha_1, & \alpha_2, & \ldots & \alpha_{n-1} \\ 0, & 1, & \alpha_1, & \ldots & \alpha_{n-2} \\ 0, & 0, & 1, & \ldots & \alpha_{n-3} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0, & 0, & 0, & \ldots & \alpha_1 \end{vmatrix}.$$

The expression for A_n as a function of the coefficients $a_1, a_2, \ldots a_n$ is deduced from the latter one in the following manner. In the equation

$$A_n = \sum_{\nu=1}^{\nu=n} \rho_n^{(\nu)} [\alpha]_\nu = \sum_{\nu=1}^{\nu=n} r_n^{(\nu)} [\alpha]_\nu,$$

let $[\alpha]_\nu$ be the complex of all terms of the dimension ν in respect to α and $[a]_\nu$ be the corresponding expression of the terms in a. Then the numerical coefficients $\rho_n^{(\nu)}$, $r_n^{(\nu)}$ are connected by the equation $r_n^{(\nu)} = (\nu + 1) \cdot \rho_n^{(\nu)}$. By this rule we find easily

$$A_1 = 2a_1;\ A_2 = 3a_1^2 - 2a_2;\ A_3 = 4a_1^3 - 6a_1 a_2 + 2a_3, \text{ etc.}$$

Next we develop the expression $\dfrac{\prod_{i=1}^{i=i} (x - \xi_i)^{\lambda_{1i} + \lambda_{2i} - 1}}{(x - \xi_\iota)^{\lambda_{1\iota} + \lambda_{2\iota} - 1}}$ into the series

$$l_0 + l_1 (x - \xi_\iota) + l_2 (x + \xi_i)^2 + \ldots + \text{ in inf.}$$

The product of this series into the series

$$A_0 - A_1 (x - \xi_\iota) + A_2 (x - \xi_\iota)^2 - \ldots \pm \text{ in inf.}$$

is equal to

$$l_0 A_0 + (l_1 A_0 - l_0 A_1) \cdot (x - \xi_\iota) + (l_2 A_0 - l_1 A_1 + l_0 A_2) \cdot (x - \xi_\iota)^2 + \ldots$$

Hence we derive the general expression for c_n in equation (8),

$$c_n = l_n \cdot A_0 - l_{n-1} \cdot A_1 + \dots + (-1)^n l_0 \cdot A_n. \tag{10}$$

This result may be enunciated in the subsequent theorem:

III. "Let ξ_ι be any of the i zeros of the integer function $\psi(x)$ in the regular differential equation

$$[\psi(x)]^2 \frac{d^2y}{dx^2} + \psi(x) \cdot F_1(x) \cdot \frac{dy}{dx} + F_2(x) \cdot y = 0$$

and $\lambda_{1\iota}$, $\lambda_{2\iota}$ the roots of the quadratic

$$[\psi'(\xi_\iota)]^2 \lambda(\lambda - 1) + \psi'(\xi_\iota) \cdot F_1(\xi_\iota) \cdot \lambda + F_2(\xi_\iota) = 0.$$

Supposing the difference $\lambda_{1\iota} - \lambda_{2\iota}$ to be equal to a positive integer n, determine a function

$$y_{1\iota} = (x - \xi_\iota)^{\lambda_{1\iota}} \cdot \{a_0 + a_1(x - \xi_\iota) + a_2(x - \xi_\iota)^2 + \dots\},$$

satisfying the proposed differential equation. Thereupon express the coefficients $A_0, A_1, \dots A_n$ in the equation

$$y_{1\iota}^{-2} \cdot (x - \xi_\iota)^{2\lambda_{1\iota}} = A_0 - A_1(x - \xi_\iota) + A_2(x - \xi_\iota)^2 \pm \dots$$

as functions of the coefficients $a_0, a_1, \dots a_n$ and likewise the coefficients $l_0, l_1, \dots l_n$ in the identity

$$\frac{\prod\limits_{i=1}^{i=\iota} (x - \xi_i)^{\lambda_{1i} + \lambda_{2i} - 1}}{(x - \xi_\iota)^{\lambda_{1\iota} + \lambda_{2\iota} - 1}} = l_0 + l_1(x - \xi_\iota) + \dots + l_n(x - \xi_\iota)^n + \dots$$

as functions of the indices

$$\lambda_{11}, \lambda_{12}, \dots \lambda_{1\iota},$$
$$\lambda_{21}, \lambda_{22}, \dots \lambda_{2\iota}.$$

If the aggregate

$$l_n \cdot A_0 - l_{n-1} A_1 + \dots + (-1)^n l_0 A_n$$

happens to be equal to zero, the point ξ_ι will be a pseudo-singular one of the given differential equation, and $y_{1\iota}$, $y_{2\iota}$ will consequently not constitute a system of fundamental integrals."

The present criterion is formally confined to finite values of ξ_ι. But the case $\xi_\iota = \xi_{\iota+1} = \infty$ may be easily reduced to the case $\xi_\iota = 0$ by changing the independent variable x in the differential equation. Indeed, if we put $x = \frac{1}{t}$ the point $t = 0$ will correspond to the point $x = \infty$.

3.

The coefficients of the differential equation (1) [No. 2] are completely determined by the indices $\lambda_{\mathfrak{p}\mathfrak{i}}$ ($\mathfrak{p} = 1, 2$; $\mathfrak{i} = 1, 2, \ldots . i, i + 1$) if i be equal to 2. In this case we may put $\xi_1 = 0$ and $\xi_1 = 1$ and obtain the differential equation

$$x^2 (x - 1)^2 \frac{d^2y}{dx^2} + x (x - 1) F_1 (x) . \frac{dy}{dx} + F_2 (x) . y = 0. \tag{1}$$

Introducing the function η instead of y, by the substitution

$$y = x^{\lambda_{11}} (x - 1)^{\lambda_{12}} . \eta,$$

η will be an integral of the equation

$$x (x - 1) \frac{d^2\eta}{dx^2} + \chi (x) . \frac{d\eta}{dx} + p . \eta = 0, \tag{2}$$

where p denotes a constant quantity. Suppose

$$\left\{ \begin{array}{ccc} 0, & 0, & \alpha \\ 1 - \gamma, & \gamma - \alpha - \beta, & \beta \end{array} \right\}.$$

to be the table of the indices of η, $\chi (x)$ will be a linear function of x satisfying the conditions

$$\chi (0) = -\gamma; \ \chi (1) = -\gamma + \alpha + \beta + 1.$$

Therefore $\chi (x) = (\alpha + \beta + 1) x - \gamma.$

From equation (1b) [No. 2] we conclude $p = \alpha\beta$. Consequently η will be a solution of the equation

$$x (x - 1) \frac{d^2\eta}{dx^2} + [(\alpha + \beta + 1) x - \gamma] \frac{d\eta}{dx} + \alpha\beta . \eta = 0. \tag{3a}$$

The complete integral-system of this well-known differential equation is

$$y_{11} = F[\alpha, \beta, \gamma, x],$$
$$y_{12} = F[\alpha, \beta, \alpha + \beta - \gamma + 1, (1 - x)],$$
$$y_{13} = x^{-\alpha} F\left[\alpha, \alpha - \gamma + 1, \alpha - \beta + 1, \frac{1}{x}\right],$$
$$y_{21} = x^{1-\gamma} . F[\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma, x],$$
$$y_{22} = (1 - x)^{\gamma - \alpha - \beta} F[\gamma - \alpha, \gamma - \beta, \gamma - \alpha - \beta + 1, (1 - x)],$$
$$y_{23} = x^{-\beta} F\left[\beta, \beta - \gamma + 1, \beta - \alpha + 1, \frac{1}{x}\right],$$

$F[\alpha, \beta, \gamma, x]$ denoting the hypergeometrical series of Gauss. This series has no definite value as soon as γ becomes a negative integer. Thus

$$F[\alpha-\gamma+1,\ \beta-\gamma+1,\ 2-\gamma,\ x] \text{ has to be rejected if } \gamma-1=n;$$

likewise $F[\gamma-\alpha,\ \gamma-\beta,\ \gamma-\alpha-\beta+1,\ (1-x)]$ if $\alpha+\beta-\gamma=n$,

and $F\left[\beta,\ \beta-\gamma+1,\ \beta-\alpha+1,\ \frac{1}{x}\right]$ if $\alpha-\beta=n$,

n denoting always a positive integer. In these cases we have the following expression for y_{21}, y_{22}, and y_{23}:

$$y_{21}=\mathfrak{P}_1(x) + c_n^{(1)}.F[\alpha, \beta, \gamma, x].\lg x \qquad (\gamma-1=n),$$

$$y_{22}=\mathfrak{P}_2(1-x)+c_n^{(2)}.F[\alpha, \beta, \alpha+\beta-\gamma+1, (1-x)].\lg(1-x) \qquad (\alpha+\beta-\gamma=n),$$

$$y_{23}=\mathfrak{P}_3\left(\frac{1}{x}\right) + c_n^{(3)}.F\left[\alpha, \alpha-\gamma+1, \alpha-\beta+1, \frac{1}{x}\right].x^{-\alpha}.\lg\frac{1}{x} \qquad (\alpha-\beta=n).$$

The condition $c_n^{(1)}=0$ denotes that $x=0$ is a pseudo-singular point of the differential equation (3a). Similarly $c_n^{(2)}=0$, $c_n^{(3)}=0$ will respectively indicate that $x=1$ or $x=\infty$ are pseudo-singular points of the hypergeometrical differential equation.

We shall pursue only the case $c_n^{(1)}=0$, as the two remaining may be absolved by following exactly the same method. The point $x=0$ will be a pseudo-singular one if the condition

$$c_n - l_n.A_0 - l_{n-1}.A_1 + l_{n-2}.A_2 - \ldots + (-1)^n l_0.A_n = 0$$

is ratified. As the coefficients $l_0, l_1, \ldots l_n$ are in the present case given by the equation

$$(1-x)^{n-\alpha-\beta} = l_0 + l_1x + l_2x^2 + \ldots + l_nx^n + \ldots,$$

we find

$$l_0=1;\ l_1=-(n-\alpha-\beta);\ l_2=\frac{(n-\alpha-\beta)(n-\alpha-\beta-1)}{1.2}, \text{ etc.}$$

Furthermore, since

$$F[\alpha, \beta, n+1, x]=1+\frac{\alpha.\beta}{n+1}.x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.(n+1)(n+2)}x^2+\ldots,$$

we get

$$A_0=1,\ A_1=\frac{2\alpha\beta}{n+1},\ A_2=\frac{\alpha^2\beta^2}{(n+1)^2}-\frac{\alpha(\alpha+1)\beta(\beta+1)}{(n+1)(n+2)},\ \ldots$$

Introducing these expressions for $l_0, l_1, \ldots. l_n$ and $A_0, A_1, \ldots. A_n$ into the equation $c_n = 0$, we obtain the latter in the form

$$c_n = (\alpha - 1)(\alpha - 2) \ldots. (\alpha - n)(\beta - 1)(\beta - 2) \ldots. (\beta - n) = 0.$$

Hence we conclude

$$\alpha = 1, 2, 3, \ldots. n \text{ and } \beta \text{ arbitrary,}$$

or

$$\beta = 1, 2, 3, \ldots. n \text{ and } \alpha \text{ arbitrary.}$$

This result justifies the theorem:

IV. "If the third element γ of the hypergeometrical series $F[\alpha, \beta, \gamma, x]$ which satisfies the differential equation

$$x(x-1)\frac{d^2\eta}{dx^2} + [(\alpha + \beta + 1)x - \gamma]\frac{d\eta}{dx} + \alpha\beta . \eta = 0$$

has the form $\gamma = n + 1$ (n being a positive integer), the expression

$$\mathfrak{P}(x) + F[\alpha, \beta, \gamma, x] . \lg x$$

together with $F[\alpha, \beta, \gamma, x]$ will constitute a complete solution of this differential equation in general. But supposing $n \geqq 1$ and α to be equal to one of the numbers $1, 2, \ldots. n$, and β remaining an arbitrary quantity, or *vice versa*, the point $x = 0$ will cease to be a singular point of the proposed differential equation."

The "relationes inter functiones contiguas" [Gauss' Werke, t. III, pag. 130] may serve to express the series $F[\alpha, \nu, n + 1, x]$ rationally by $F[\alpha, 1, 2, x]$ and $F[\alpha, 1, 3, x]$. By means of the formulas

$$\begin{aligned}
&(\nu - 1)(1 - x) F[\alpha, \nu, n + 1, x] \\
&+ [n - 2\nu - 1 - (\alpha - \nu + 1)x] . F[\alpha, \nu - 1, n + 1, x] \\
&- (n - \nu + 2) F[\alpha, \nu - 2, n + 1, x] = 0 \qquad \text{(form. (1) l. c.),} \\
&(n - \alpha) . x . F[\alpha, \nu, n + 1, x] \\
&+ n[n - 1 - (2n - \alpha - \nu - 1)x] . F[\alpha, \nu, n, x] \\
&- n(n - 1)(1 - x) . F[\alpha, \nu, n - 1, x] \qquad \text{(form. (15) l. c.)}
\end{aligned}$$

$F[\alpha, \nu, n + 1, x]$ is expressible by

$$F[\alpha, 1, 2, x],\ F[\alpha, 1, 3, x],\ F[\alpha, 2, 2, x],\ F[\alpha, 2, 3, x].$$

But between these four functions there exist two relations:

$$[1-(2-\alpha)x].F[\alpha,\,2,\,3,\,x]+F[\alpha,\,1,\,3,\,x]-2(1-x).F[\alpha,\,2,\,2,\,x]=0$$

(cf. form. (14) l. c.),

$$2[1-(2-\alpha)x].F[\alpha,\,1,\,2,\,x]-2(1-x).F[\alpha,\,2,\,2,\,x] + (2-\alpha)x.F[\alpha,\,1,\,3,\,x]=0$$

(cf. form. (11) l. c.).

Therefore $F[\alpha,\,\nu,\,n+1,\,x]$ can be reduced to $F[\alpha,\,1,\,2,\,x]$ and $F[\alpha,\,1,\,3,\,x]$. We find easily

$$F[\alpha,\,1,\,2,\,x]=\frac{1}{1-\alpha}\cdot\frac{1}{x}\cdot[1-(1-x)^{1-\alpha}],$$

$$F[\alpha,\,1,\,3,\,x]=\frac{2}{1-\alpha}\cdot\frac{1}{x}-\frac{2}{(2-\alpha)(1-\alpha)}\cdot\frac{1}{x^2}[1-(1-x)^{2-\alpha}].$$

This is sufficient to show that $F[\alpha,\,\nu,\,n+1,\,x]$ is "a rational function of x and $(1-x)^{\alpha}$" $[1\leqq\nu\leqq n]$.

4.

The general theory of differential equations of the second order with more than two finite singular points is very little elaborated up to the present. Only a few special cases have been treated by Green and Lamé, whose researches have been completed and enlarged by Liouville, Heine, Hermite, Fuchs, and other mathematicians.

Let us first consider differential equations with a single characteristic parameter. This will occur for $i=3$. Equation (1) [No. 2] assumes in the present case ($\xi_1=0,\ \xi_2=1,\ \xi_3=a$) the form

$$x^2(x-1)^2(x-a)^2\frac{d^2y}{dx^2}+x(x-1)(x-a).F_1(x)\frac{dy}{dx}+F_2(x).y=0. \qquad (1)$$

This equation is satisfied by a function y with the indices

$$\begin{array}{cccc} 0. & 2. & a. & \infty. \\ \hline \lambda_{11}, & \lambda_{12}, & \lambda_{13}, & \lambda_{14}, \\ \lambda_{21}, & \lambda_{22}, & \lambda_{23}, & \lambda_{24}. \end{array}$$

If we put

$$y=x^{\lambda_{11}}(x-1)^{\lambda_{12}}(x-a)^{\lambda_{13}}.\eta,$$

the new function η will be an integral of the equation

$$x(x-1)(x-a)\frac{d^2\eta}{dx^2}+\chi(x).\frac{d\eta}{dx}+\varpi(x).\eta=0. \qquad (2)$$

Suppose the table of the indices of η to be

$$\left\{\begin{array}{cccc} \underline{0.} & \underline{1.} & \underline{a.} & \underline{\infty.} \\ 0, & 0, & 0, & \alpha \\ 1-\gamma, & 1-\delta, & \gamma-\alpha+\delta-\beta, & \beta \end{array}\right\}$$

By equation (2a) [No. 2] we have therefore

$$\chi(0)=\psi'(0).\gamma,\ \chi(1)=\psi'(1).\delta,\ \chi(a)=-\psi'(a)[\gamma-\alpha+\delta-\beta-1],$$

or since $\psi(x)=x(x-1)(x-a)$, these equations are

$$\chi(0)=a\gamma,\ \chi(1)=(1-a)\delta,\ \chi(a)=-a(a-1)[\gamma-\alpha+\delta-\beta-1],$$

consequently

$$\chi(x)=(\alpha+\beta+1)x^2-\{\alpha+\beta-\delta+1+a(\gamma+\delta)\}.x+a\gamma.$$

Equation (1b) [No. 2] gives for $i=3$,

$$\tilde{\omega}(x)=\alpha\beta.x+\varkappa_1.$$

Writing $-\alpha\beta g$ instead of $\varkappa_1$, the differential equation (2) will take the final form

$$\left.\begin{array}{l} x(x-1)(x-a)\dfrac{d^2\eta}{dx^2} \\ +\left[(\alpha+\beta+1)x^2-\{\alpha+\beta-\delta+1+a(\gamma+\delta)\}x+a\gamma\right]\dfrac{d\eta}{dx} \\ +\alpha\beta(x-g)\eta=0. \end{array}\right\} \qquad (2a)$$

The quantity g represents the characteristic parameter of this equation. Mod. (a) may be supposed to be >1. Should it be <1, then it will be easy to transform equation (2a) into a similar one for which the first hypothesis holds good. Now let us determine the coefficients $a_1, a_2 \ldots.$ in the series

$$\eta=\sum_{\nu=0}^{\nu=\infty} a_\nu.x^\nu, \qquad (3)$$

that η be a particular solution of the equation (2a). The latter equation will be identically satisfied by the series (3) if

$$\begin{array}{l} \displaystyle\sum_{\nu=0}^{\nu=\infty} a_\nu\left[\{\nu(\nu-1)+(\alpha+\beta+1)\nu+\alpha\beta\}x^{\nu+1}\right. \\ \qquad -\{\nu(\nu-1)(a+1)+\nu[\alpha+\beta-\delta+1+a(\gamma+\delta)]+\alpha\beta g\}x^\nu \\ \qquad \left.+\{\nu(\nu-1)a+\nu a\gamma\}x^{\nu-1}\right]=0. \end{array}$$

From this identity we derive formula

$$\left.\begin{array}{l}(\nu+\alpha-1)(\nu+\beta-1)a_{\nu-1}-\{\nu[\nu+\alpha+\beta-\delta+a(\nu+\gamma+\delta-1)]+\alpha\beta g\}\,a_\nu \\ \qquad +a(\nu+1)(\nu+\gamma)a_{\nu+1}=0.\end{array}\right\}\quad(4)$$

a_1 and a_0 are connected by the equation $a\gamma a_1-\alpha\beta g a_0=0$. Equation (4) gives

$$a_\nu=\frac{1}{1.2\ldots.\nu}\cdot\frac{\alpha.\beta}{\gamma(\gamma+1)\ldots.(\gamma+\nu-1)}\cdot\frac{G^{(\nu)}_{(g)}}{a^\nu}\qquad(5)$$

where $G^{(\nu)}_{(g)}$ denotes a certain integer function of g of the degree ν. If we put

$$\left.\begin{array}{l}F[a(\alpha,\beta,\gamma,\delta)g,x] \\ \quad =1+\dfrac{\alpha\beta}{\gamma}\left\{G^{(1)}_{(g)}\cdot\dfrac{x}{a}+\dfrac{G^{(2)}_{(g)}}{1.2.\gamma(\gamma+1)}\cdot\dfrac{x^2}{a^2}+\dfrac{G^{(3)}_{(g)}}{1.2.3.\gamma(\gamma+1)(\gamma+2)}\cdot\dfrac{x^3}{a^3}\right. \\ \qquad \left.+\dfrac{G^{(4)}_{(g)}}{1.2.3.4.\gamma(\gamma+1)(\gamma+2)(\gamma+3)}\cdot\dfrac{x^4}{a^4}+\ldots.+\text{in inf.}\right.\end{array}\right\}\quad(6)$$

$F[a(\alpha,\beta,\gamma,\delta)g,x]$ will be certainly an integral function of the differential equation (2a) if Mod. $(x)<1$. Substituting a_ν from equation (5) into equation (4) we obtain for the successive determination of the functions $G^{(3)}_{(g)}$, $G^{(4)}_{(g)}$, ..., the formula

$$\left.\begin{array}{l}G^{(\nu+1)}_{(g)}=\{\nu[\nu+\alpha+\beta-\delta+a(\nu+\gamma+\delta)]+\alpha\beta g\}\,G^{(\nu)}_{(g)} \\ \qquad -a\nu(\alpha+\nu-1)(\beta+\nu-1)(\gamma+\nu-1).\,G^{(\nu-1)}_{(g)}=0,\end{array}\right\}\quad(7)$$

starting with the functions

$$G^{(1)}_{(g)}=g,\ G^{(2)}_{(g)}=\alpha\beta g^2+\{\alpha+\beta-\delta+1+a(\gamma+\delta)\}g-a\gamma.$$

In the special case $a=1$, $g=1$, we have

$$G^{(\nu)}_{(1)}=(\alpha+1)(\alpha+2)\ldots.(\alpha+\nu-1).(\beta+1)(\beta+2)\ldots.(\beta+\nu-1),$$

consequently

$$F[1(\alpha,\beta,\gamma,\delta)1,x]=F[\alpha,\beta,\gamma,x].$$

This relation is important, as it shows that Gauss' hypergeometrical series may be looked at as a "degeneration" of our general series $F[a(\alpha,\beta,\gamma,\delta)g,x]$.

Since the functions of Green* and Lamé† are solutions of the differential equation

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* "On the determination of the exterior and interior attractions," etc., in the Transactions of the Cambridge Phil. Soc., 1835 (read 6th May, 1833). This paper contains already the fundamental properties of the functions which have been afterwards called by Heine "höhere Lamésche Functionen."

† Liouville's Journal, t. IV (1839) and "Leçons sur les fonctions inverses," Paris, 1857.

$$x(x-1)(x-a)\frac{d^2w}{dx^2}+\frac{1}{2}\left[3x^2-2(a+1)x+a\right]\frac{dw}{dx}-\frac{1}{4}n(n+1)(x+g)w=0.$$

n being a positive integer, we have the expression

$$w=F\left[a\left(-\frac{1}{2}n,\,\frac{1}{2}(n+1),\,\frac{1}{2},\,\frac{1}{2}\right)g,\,x\right].$$

A second solution of the differential equation (2a) must necessarily have the form $\eta_{21}=x^{1-\gamma}.\mathfrak{P}(x)$.

Now an elementary transformation of the indices gives the relation*

$$\eta\left\{\begin{matrix}0, & 0, & 0, & \alpha\\ 1-\gamma, & 1-\delta, & \gamma-\alpha+\delta-\beta, & \beta\end{matrix}\right\}$$

$$=x^{1-\gamma}.\eta\left\{\begin{matrix}\gamma-1, & 0, & 0, & \alpha-\gamma+1\\ 0, & 1-\delta, & \gamma-\alpha+\delta-\beta, & \beta-\gamma+1\end{matrix}\right\}$$

$$=x^{1-\gamma}.\eta\left\{\begin{matrix}0, & 0, & 0, & \alpha-\gamma+1\\ 1-(2-\gamma), & 1-\delta, & \gamma-\alpha+\delta-\beta, & \beta-\gamma+1\end{matrix}\right\}$$

Hence follows immediately

$$y_{21}=x^{1-\gamma}.F[a(\alpha-\gamma+1,\ \beta-\gamma+1,\ 2-\gamma,\ \delta)g,\,x]. \tag{8}$$

But if $\gamma=n+1$ (n being a positive integer), this second solution has to be rejected and substituted by

$$y_{21}=\mathfrak{P}_2(x)+c_n.F[a(\alpha,\beta,\gamma,\delta)g,\,x].\lg x.$$

The point $x=0$ will be a pseudo-singular one if the condition $c_n=0$ be satisfied. It is evident that A_n contains no a_ν whose index ν is $>n$. Therefore A_n is an integer function of g of the degree n. The coefficients $l_0, l_1, \ldots . l_n$ are in the present case determined by the equation

$$(x-1)^{-\delta}(x-a)^{n-\alpha+\delta-\beta}=l_0+l_1x+l_2x^2+\ldots .+l_nx^n+\ldots .$$

They are of course independent of g. Consequently the condition $c_n=0$ will become an algebraical equation of the n^{th} degree in respect to the quantity g. Hence the theorem:

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* This manner of denoting integral functions by their table of the indices is adopted from Riemann.

V. "If, in the differential equation

$$x(x-1)(x-a)\frac{d^2\eta}{dx^2}+[(\alpha+\beta+1)x^2-\{\alpha+\beta-\delta+1+a(\gamma+\delta)\}x+a\gamma]\frac{d\eta}{dx}, \\ +\alpha\beta(x-g).\eta=0,$$

of which $F[a(\alpha,\beta,\gamma,\delta)g,x]$ is a solution, the third element γ takes the value $\gamma=n+1$ (n being a positive integer), then it is always possible to determine the characteristic parameter g by an algebraical equation of the degree n, so that the point $x=0$ degenerates into a pseudo-singular one. The elements α,β,δ are not subjected to any algebraical condition."

Theorems altogether analogous to the preceding one may be enunciated in reference to the points $x=1,\ a,\ \infty$.

5.

Let us proceed to differential equations of the second order with any number of singular points. Suppose $\xi_1,\xi_2,\ldots\xi_0$ of the series $\xi_1,\xi_2,\ldots\xi_i$ to have degenerated into pseudo-singular points. This hypothesis implies the equation (cf. Theorem III)

$$c_{n_1}=0,\ c_{n_2}=0,\ \ldots c_{n_v}=0, \tag{1}$$

and subjects, therefore, the $i-2$ characteristic parameters of the equation

$$\psi(x)\frac{d^2\eta}{dx^2}+\chi(x)\frac{d\eta}{dx}+[\mathfrak{l}^{(1)}_{i+1}.\mathfrak{l}^{(2)}_{i+1}.x^{i-2}+\varkappa_1x^{i-3}+\ldots+\varkappa_{i-2}].\eta=0 \tag{2}$$

to v conditional equations.

The original table of the indices of η is

$$\begin{Bmatrix}0, & 0, & \ldots & 0, & \mathfrak{l}^{(1)}_{i+1}\\ \mathfrak{l}_1, & \mathfrak{l}_2, & \ldots & \mathfrak{l}_i, & \mathfrak{l}^{(2)}_{i+1}\end{Bmatrix}.$$

As soon as

$$\begin{aligned}\mathfrak{l}_1&=-n_1, & \mathfrak{l}_2&=-n_2, & \ldots\mathfrak{l}_v&=-n_v,\\ c_{n_1}&=0, & c_{n_2}&=0, & \ldots c_n&=0,\end{aligned}$$

only $\xi_{v+1},\xi_{v+2},\ldots\xi_i$ remain proper singular points of equation (2), which we write now in the form

$$\phi(x).\psi_1(x)\frac{d^2\eta}{dx^2}+\chi(x).\frac{d\eta}{dx}+\tilde{\omega}(x),\ \eta=0, \tag{3}$$

making $\phi(x)=(x-\xi_1)(x-\xi_2)\ldots(x-\xi_v)$ and $\psi_1(x)=\dfrac{\psi(x)}{\phi(x)}$.

The last differential equation has not the regular form (A) [No. 1], notwithstanding its general integral η may be expressed by the general integral z of the auxiliary regular differential equation

$$\psi_1(x)\frac{d^2z}{dx^2}+\chi_1(x)\frac{dz}{dx}+\tilde{\omega}_1(x).z=0, \tag{4}$$

where $\chi_1(x)$ and $\tilde{\omega}_1(x)$ denote integer functions of the degrees $i-v-1$, $i-v-2$. The table of the indices of the new function z is supposed to be

$$\left\{\begin{matrix} 0 & , 0 & , \ldots . 0, & \mathfrak{l}_{i+1}^{(1)} \\ \mathfrak{l}_{v+1}, & \mathfrak{l}_{v+2}, & \ldots . \mathfrak{l}_i, & \mathfrak{l}_{i+1}^{(2)}-\sum\limits_{\mathfrak{i}=1}^{\mathfrak{i}=v} n_i \end{matrix}\right\}$$

The general theory of linear differential equations requires the sum of all the indices in this table to be equal to $i-1$ (cf. Fuchs, Journ. für Mathematik, t. 66, pag. 146). In fact we have

$$\sum_{\mathfrak{i}=1}^{\mathfrak{i}=i}\mathfrak{l}_{\mathfrak{i}}+\mathfrak{l}_{i+1}^{(1)}+\mathfrak{l}_{i+1}^{(2)}=i-1,$$

but $\sum\limits_{\mathfrak{i}=1}^{\mathfrak{i}=v}\mathfrak{l}_{\mathfrak{i}}=\sum\limits_{\mathfrak{i}=1}^{\mathfrak{i}=v} n_{\mathfrak{i}}$, consequently $\sum\limits_{\mathfrak{i}=\nu+1}^{\mathfrak{i}=i}\mathfrak{l}_{\mathfrak{i}}+\mathfrak{l}_{i+1}^{(1)}+\mathfrak{l}_{i+1}^{(2)}-\sum\limits_{\mathfrak{i}=1}^{\mathfrak{i}=v} n_{\mathfrak{i}}=i-2$, q. e. d.

The function $\chi_1(x)$ is completely determined by the relations

$$\chi(\xi_{\mathfrak{i}})=-\psi'(\xi_{\mathfrak{i}})(\mathfrak{l}_{\mathfrak{i}}-1)\,|\,\mathfrak{i}=v+1,\ v+2,\ \ldots . i.$$

$\tilde{\omega}_1(x)$ must necessarily be of the form

$$\tilde{\omega}_1(x)=\mathfrak{l}_{\mathfrak{i}+1}^{(1)}\left(\mathfrak{l}_{i+1}^{(2)}-\sum_{\mathfrak{i}=1}^{\mathfrak{i}=v} n_{\mathfrak{i}}\right).x^{i-v-2}+\varkappa_1'.x^{i-v-3}+\ldots .+\varkappa_{i-v-2}',$$

$\varkappa_1', \varkappa_2', \ldots . \varkappa_{i-v-2}'$ being the characteristic parameters of the differential equation (4).

Now it is always possible to determine two integer functions $P_0(x)$ and $P_1(x)$ and to express $\varkappa_1', \varkappa_2', \ldots . \varkappa_{i-v-2}'$ by means of $\varkappa_1, \varkappa_2, \ldots . \varkappa_{i-2}$ that the relation

$$\eta=P_0(x).z+P_1(x).\frac{dz}{dx}$$

exists between the functions η and z. The particulars of this method of reduction are fully explained in my paper quoted in the beginning of No. 1 (Acta Mathem. t. 11, pag. 97).

The preceding remarks lead to the subsequent theorem:

VI. "Whenever v of the i zeros $(\xi_1, \xi_2, \ldots \xi_i)$ of the integral function $\psi(x)$ in the differential equation

$$\psi(x)\cdot\frac{d^2\eta}{dx^2}+\chi(x)\cdot\frac{d\eta}{dx}+\tilde{\omega}(x)\cdot\eta=0$$

degenerates into pseudo-singular points, the function η will be expressible rationally by a function z satisfying the regular differential equation

$$\psi_1(x)\,\frac{d^2z}{dx^2}+\chi_1(x)\cdot\frac{dz}{dx}+\tilde{\omega}_1(x)\cdot z=0$$

where the zeros of $\psi_1(x)$ are the proper singular points of η and its first derivative function $\frac{dz}{dx}$. The $i-v-2$ characteristic parameters of z are known functions of the $i-2$ characteristic parameters of η."

Hence we conclude, if $x=0$ is a pseudo-singular point of the differential equation (2a) [No. 4], the function $F[a(\alpha, \beta, n+1, \delta)g, x]$ will be expressible by the hypergeometrical series of Gauss.

FRANKFURT A.M., 27 Dec., 1887.

P. S.—M. Poincaré has requested me to correct an error in his memoir, "Sur les groupes des équations linéaires," Acta Math., Vol. IV, p. 217: "Pour qu'un infini des coëfficients ϕ_κ soit un point à apparence singulière, il faut $\frac{(p+2)(p-1)}{2}$ conditions"—this should be ".... $2p-2$ conditions." K. H.

On some Applications of the Units of an n-fold Space.

By C. H. Chapman.

The following article originated in an attempt to obtain a proof of the rule for multiplying two determinants of the n^{th} order by the principles of quaternions. There is no difficulty, of course, in the case of a determinant of the third order, but beyond that it seemed necessary to use a more general system of vector units than Hamilton's i, j, k. The symmetry of an n-dimensional space, where n is odd, leads at once to the proper assumptions, as has been often shown. In working with n rectangular unit vectors the symbols S and V as defined by Hamilton were found not sufficiently exclusive, and, as a makeshift, a new symbol ω_κ was defined and found extremely useful. When applied to the product of two vectors ω_κ has the same selective power as V, but applied to the product of three vectors, it selects the same terms as S. Applied to a product of more than three vectors ω_κ is equivalent neither to S nor V.

Aided by this symbol, the rules for determinants of the n^{th} order were found with ease and a method of inverting the linear and vector function ϕ in space of n dimensions was generalized from that given by Hamilton for ordinary space. The fact that $\phi\rho$ effects a linear substitution upon the coefficients of ρ leads on to the interesting connection between the operator ϕ and the theory of linear differential equations, and by means of its properties a number of theorems may be demonstrated with facility.

I begin by showing the connection between ordinary quaternions and determinants of the third order.

If α, β, γ and α', β', γ' be two systems of non-coplanar vectors in space of three dimensions, it is easy to obtain the following equation, viz:

$$S\alpha\beta\gamma\,.\,S^3\alpha'\beta'\gamma' = S\,.\,V\alpha'\beta'\,V\beta'\gamma'\,V\gamma'\alpha' \begin{vmatrix} S\alpha'\alpha\,, & S\beta'\alpha\,, & S\gamma'\alpha \\ S\alpha'\beta\,, & S\beta'\beta\,, & S\gamma'\beta \\ S\alpha'\gamma\,, & S\beta'\gamma\,, & S\gamma'\gamma \end{vmatrix}$$

But plainly,

$$\left.\begin{aligned} S.V\alpha'\beta'V\beta'\gamma'V\gamma'\alpha' &= S.V\alpha'\beta'V.V\beta'\gamma'V\gamma'\alpha' \\ &= -S.V\alpha'\beta'.\gamma'S\alpha'\beta'\gamma' = -S^2\alpha'\beta'\gamma'. \end{aligned}\right\} \quad (1)$$

Hence

$$S\alpha\beta\gamma.S\alpha'\beta'\gamma' = -\begin{vmatrix} S\alpha'\alpha, & S\beta'\alpha, & S\gamma'\alpha \\ S\alpha'\beta, & S\beta'\beta, & S\gamma'\beta \\ S\alpha'\gamma, & S\beta'\gamma, & S\gamma'\gamma \end{vmatrix} \quad (2)$$

If now

$$\alpha = x_1 i + y_1 j + z_1 k,\ \beta = x_2 i + y_2 j + z_2 k,\ \gamma = x_3 i + y_3 j + z_3 k,$$
$$\alpha' = x_1' i + y_1' j + z_1' k,\ \beta' = x_2' i + y_2' j + z_2' k,\ \gamma' = x_3' i + y_3' j + z_3' k,$$

then

$$S\alpha\beta\gamma = -\begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix},\ -S\alpha'\beta'\gamma' = \begin{vmatrix} x_1', & y_1', & z_1' \\ x_2', & y_2', & z_2' \\ x_3', & y_3', & z_3' \end{vmatrix}$$

and

$$\begin{aligned} S\alpha'\alpha &= -(x_1x_1' + y_1y_1' + z_1z_1'), \\ S\beta'\alpha &= -(x_1x_2' + y_1y_2' + z_1z_2'), \\ S\gamma'\alpha &= -(x_1x_3' + y_1y_3' + z_1z_3'), \end{aligned}$$

and so on for the other scalars. Hence

$$\begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix} \times \begin{vmatrix} x_1', & y_1', & z_1' \\ x_2', & y_2', & z_2' \\ x_3', & y_3', & z_3' \end{vmatrix} = \begin{vmatrix} x_1x_1'+y_1y_1'+z_1z_1', & x_1x_2'+y_1y_2'+z_1z_2', & x_1x_3'+y_1y_3'+z_1z_3' \\ x_2x_1'+y_2y_1'+z_2z_1', & x_2x_2'+y_2y_2'+z_2z_2', & x_2x_3'+y_2y_3'+z_2z_3' \\ x_3x_1'+y_3y_1'+z_3z_1', & x_3x_2'+y_3y_2'+z_3z_2', & x_3x_3'+y_3y_3'+z_3z_3' \end{vmatrix}$$

by eq. (2). Giving thus the rule for multiplying determinants of the third order. Noting that $V\alpha\beta = (x_1y_2 - x_2y_1)\,ij + \ldots\ldots = (x_1y_2 - x_2y_1)\,k + \ldots\ldots$, eq. (1) gives

$$\begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix}^2 = \begin{vmatrix} x_1y_2 - y_1x_2, & z_1x_2 - x_1z_2, & y_1z_2 - z_1y_2 \\ x_2y_3 - y_2x_3, & z_2x_3 - x_2z_3, & y_2z_3 - z_2y_3 \\ x_3y_1 - y_3x_1, & z_3x_1 - x_3z_1, & y_3z_1 - z_3y_1 \end{vmatrix}$$

or the square of a determinant of the third order is equal to the determinant of its minors.

To generalize these results I consider a system of n units like i, j, k in a space of n dimensions, where n is an odd integer. Let these units be $\iota_1, \iota_2, \iota_3, \ldots\ldots \iota_n$, and let $S\iota_\kappa\iota_\tau = 0$; also $\iota_\kappa\iota_\tau = -\iota_\tau\iota_\kappa$. Also the complete cycle when multiplied together produces a scalar. Assume then

$$\iota_1\iota_2 \ldots\ldots \iota_{n-1}\iota_n = A \quad (3)$$

where $A = \pm 1$, as will be shown.

Again, since $\iota^2 = \iota\iota = -1$, it follows that

$$\iota_\kappa = -\frac{1}{\iota_\kappa} = -\iota_\kappa^{-1}.$$

To determine the rule for the essential sign of A, note that, by eqs. (3) and (4),

$$l_1 l_2 \dots l_{n-1} = -A l_n.$$

Also by successive multiplications

$$l_n = A l_{n-1}^{-1} l_{n-2}^{-1} \dots l_1^{-1} = (-1)^{n-1} A l_{n-1} l_{n-2} \dots l_1 = A l_{n-1} l_{n-2} \dots l_1,$$

since n is odd. By the continued use of eq. (4) this becomes

$$l_n = (-1)^{\frac{(n-1)(n-2)}{2}} A . l_1 l_2 l_3 \dots l_{n-1},$$

or

$$l_1 l_2 \dots l_{n-1} = (-1)^{\frac{(n-1)(n-2)}{2}} \frac{1}{A};$$

hence

$$-A = (-1)^{\frac{(n-1)(n-2)}{2}} \frac{1}{A},$$

or

$$A = (-1)^{\frac{n(n-1)}{2}} \frac{1}{A}. \tag{5}$$

This is satisfied only if $A = (-1)^{\frac{n(n-1)}{2}}$. From the symmetry of the space it is safe to conclude that in general

$$l_\kappa l_{\kappa+1} \dots l_n l_1 l_2 \dots l_{\kappa-1} = (-1)^{\frac{n(n-1)}{2}} = A,$$

or if proof were needed, multiplying l_1 by the first member of eq. (3), then multiplying the result by l_1, we have, if the equation be preserved,

$$l_1 l_1 l_2 l_3 \dots l_n l_1 = A l_1 l_1,$$

or

$$-l_2 l_3 \dots l_n l_1 = -A,$$

or

$$l_2 l_3 \dots l_n l_1 = A,$$

a process which may be carried on indefinitely. Thus both the sign and numerical value of A are consistently determined by eq. (5) for the whole series of cyclic products formed from the n units.

To determine what the value of $l_1 l_2$ is, since $S l_1 l_2 = 0$, it may be assumed at once that

$$l_1 l_2 = x_1 l_1 + x_2 l_2 + x_3 l_3 + \dots + x_n l_n.$$

Then

$$S l_1 l_1 l_2 = 0 = -x_1 + x_2 S l_1 l_2 + \dots = -x_1.$$

Likewise

$$x_2 = 0 \text{ and } x_3 = -S l_1 l_2 l_3,\ x_4 = -S l_1 l_2 l_4, \dots$$

Hence the product of two vectors is a linear function of the remaining $n-2$ of the system.

Let it be required to express a vector ρ linearly in terms of the fundamental units $l_1 l_2 \dots l_n$.

Assume the equation

$$\rho = x_1 l_1 + x_2 l_2 + \ldots . + x_n l_n .$$

Then at once, multiplying by l_μ and taking scalars,

$$x_\mu = - S l_\mu \rho , \tag{6}$$

which determines the coefficients. If

$$\rho_1 = y_1 l_1 + y_2 l_2 + \ldots . + y_n l_n ,$$

since $l_\kappa l_\tau = - l_\tau l_\kappa$,

$$V \rho \rho_1 = - V \rho_1 \rho , \tag{7}$$

and plainly also, $S\rho\rho_1 = S\rho_1\rho$.

Let $\alpha_1, \alpha_2, \ldots . \alpha_n$ be a system of n vectors such that

$$\left. \begin{array}{l} \alpha_1 = x_{11} l_1 + x_{12} l_2 + x_{13} l_3 + \ldots . + x_{1n} l_n , \\ \alpha_2 = x_{21} l_1 + x_{22} l_2 + x_{23} l_3 + \ldots . + x_{2n} l_n , \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \alpha_n = x_{n1} l_1 + x_{n2} l_2 + x_{n3} l_3 + \ldots . + x_{nn} l_n . \end{array} \right\} \tag{A}$$

Let ω_κ be a symbol such that when applied to the product of any κ vectors it shall select all those terms containing the product of any κ distinct units and shall exclude all other terms, the selected terms being united into an aggregate by addition.

Let $\omega_\kappa \alpha_1 \alpha_2 \ldots . \alpha_\kappa$, or briefly $\omega_\kappa(\alpha_1)$, when the cyclic order is perfect, be such an aggregate. Considering any two vectors $\alpha_1 \alpha_2$, it is clear that only those terms of their product included in $\omega_2 \alpha_1 \alpha_2$ enter into ω_κ. Hence the symbol ω_2 may be arbitrarily inserted in a product under ω_κ. And in general, if $\lambda < \kappa$, to insert ω_λ under ω_κ before any group of λ vectors is merely to bring into evidence an operation implied in ω_κ itself. Therefore

$$\omega_\kappa \alpha_1 \ldots . \alpha_\kappa = \omega_\kappa \alpha_1 \alpha_2 \ldots . \alpha_\mu \left[\omega_\lambda \alpha_{\mu+1} \ldots . \right] \alpha_{\mu+\lambda+1} \ldots . \alpha_\kappa .$$

ω_κ being merely selective, is plainly commutative with all scalars and distributive over a sum.

$\omega_\kappa(\alpha_1)$ is the sum of all the determinants in the matrix

$$\left| \begin{array}{l} x_{11}, \ x_{12}, \ x_{13}, \ \ldots . \ x_{1n} \\ x_{21}, \ x_{22}, \ x_{23}, \ \ldots . \ x_{2n} \\ \ldots\ldots\ldots\ldots\ldots \\ x_{\kappa 1}, \ x_{\kappa 2}, \ x_{\kappa 3}, \ \ldots . \ x_{\kappa n} \end{array} \right| ,$$

each multiplied by the group of units whose suffixes are determined by the second suffixes of the principal diagonal. So far as the letters in each product are concerned, this is only another statement of the definition of ω_κ; and as to the signs, it need only be noticed that given any group of $\varkappa$ units,

$$(M),\ l_1 l_3 l_4 l_5 \dots\dots l_\tau l_\mu$$

where no two are identical, then any arrangement produced from (M) by λ interchanges of consecutive vectors equals $(-1)^\lambda(M)$, so that the signs of the various terms are precisely those required. Hence, taking $\varkappa = n$, $\omega_n(\alpha_1 \dots\dots \alpha_n)$

$$= \begin{vmatrix} x_{11}, & x_{12}, & \dots\dots & x_{1n} \\ x_{21}, & x_{22}, & \dots\dots & x_{2n} \\ \dots & \dots & \dots & \dots \\ x_{n1}, & x_{n2}, & \dots\dots & x_{nn} \end{vmatrix} A,$$

where $A = l_1 \dots\dots l_n$.

When dealing with only two vectors, ω_2 is the same as V, as a glance at the system (A) will show; and $\omega_{n-1}\alpha_1 \dots\dots \alpha_{n-1}$ is a vector, since the product of any $n-1$ units produces the remaining one multiplied by $\pm A$; but

$$\omega_{n-1}(\alpha_1 \dots\dots \alpha_{n-1}) \neq V.\alpha_1 \dots\dots \alpha_{n-1}.$$

Also, while $\omega_n\alpha_1 \dots\dots \alpha_n$ is a scalar, it is not the same as $S.\alpha_1 \dots\dots \alpha_n$.

From eq. (7), then, $\omega_2\rho_1\rho = -\omega_2\rho\rho_1$. Hence

$$\begin{aligned} \omega_n\alpha_1 \dots\dots \alpha_{\kappa-1}\alpha_\kappa \dots\dots \alpha_n &= \omega_n\alpha_1 \dots\dots (\omega_2\alpha_{\kappa-1}\alpha_\kappa) \dots\dots \alpha_n \\ &= -\omega_n\alpha_1 \dots\dots (\omega_2\alpha_\kappa\alpha_{\kappa-1}) \dots\dots \alpha_n \\ &= -\omega_n\alpha_1 \dots\dots \alpha_\kappa\alpha_{\kappa-1} \dots\dots \alpha_n. \end{aligned}$$

Therefore, to interchange two rows (or columns) changes the sign of the determinant.

Again, $\omega_n\alpha_1 \dots\dots \alpha_\kappa\alpha_\kappa \dots\dots \alpha_{n-1} = \omega_n\alpha_1 \dots\dots (\omega_2\alpha_\kappa\alpha_\kappa) \dots\dots \alpha_{n-1}$. But $\omega_2\alpha_\kappa\alpha_\kappa = 0$ evidently. Hence $\omega_n(\alpha) = 0$ if it contains the same vector twice, and a determinant vanishes if two of its rows are identical.

If one of the n vectors $\alpha_1 \dots\dots \alpha_\mu \dots\dots \alpha_n$ is a linear function of the others, we have

$$\omega_n\alpha_1 \dots\dots \alpha_\mu \dots\dots \alpha_n = \omega_n\alpha_1\alpha_2 \dots\dots (c_1\alpha_1 + c_2\alpha_2 + \dots\dots + c_n\alpha_n) \dots\dots \alpha_n = 0,$$

since every product will contain some vector twice. Also

$$\omega_n\alpha_1\alpha_2 \dots\dots (\alpha_\mu + c_1\alpha_1 + c_2\alpha_2 + \dots\dots) \dots\dots \alpha_n = \omega_n\alpha_1 \dots\dots \alpha_n,$$

since the products formed with the added vectors all vanish. Hence a determinant is not changed by adding to the terms of any row the corresponding terms of any other row multiplied by a common multiplier.

Consider the system

$$\begin{aligned} \beta_1 &= y_{11}\alpha_1 + y_{12}\alpha_2 + \dots\dots + y_{1n}\alpha_n, \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ \beta_n &= y_{n1}\alpha_1 = y_{n2}\alpha_2 + \dots\dots + y_{nn}\alpha_n. \end{aligned}$$

From what has already been shown it is clear that

$$\omega_n\beta_1\beta_2 \dots\dots \beta_n = \begin{vmatrix} y_{11}, & y_{12}, & y_{13} & \dots\dots & y_{1n} \\ y_{21}, & y_{22}, & y_{23} & \dots\dots & y_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ y_{n1}, & y_{n2}, & y_{n3} & \dots\dots & y_{nn} \end{vmatrix} \omega_n\alpha_1 \dots\dots \alpha_n,$$

which again

$$= \begin{vmatrix} y_{11}, & y_{12} & \dots\dots & y_{1n} \\ \dots & \dots & \dots & \dots \\ y_{n1}, & y_{n2} & \dots\dots & y_{nn} \end{vmatrix} \times \begin{vmatrix} x_{11}, & x_{12} & \dots\dots & x_{1n} \\ \dots & \dots & \dots & \dots \\ x_{n1}, & x_{n2} & \dots\dots & x_{nn} \end{vmatrix} A.$$

But

$$\begin{aligned} \beta_1 = (y_{11}x_{11} + y_{12}x_{21} + y_{13}x_{31} + \dots\dots + y_{1n}x_{n1})\, l_1 \\ + (y_{11}x_{12} + y_{12}x_{22} + y_{13}x_{32} + \dots\dots + y_{1n}x_{n2})\, l_2 + \dots\dots, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \beta_\mu = (y_{\mu1}x_{11} + y_{\mu2}x_{21} + \dots\dots)\, l_1 + (y_{\mu1}x_{12} + y_{\mu2}x_{22} + \dots\dots)\, l_2 + \dots\dots \end{aligned}$$

Hence $\omega_n\beta_1\beta_2 \dots\dots \beta_n$

$$= \begin{vmatrix} y_{11}x_{11} + y_{12}x_{21} + \dots\dots + y_{1n}x_{n1}, & y_{11}x_{12} + y_{12}x_{22} + \dots\dots + y_{1n}x_{n2}, & \dots\dots, & \dots\dots \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ y_{n1}x_{11} + y_{n2}x_{21} + \dots\dots + y_{nn}x_{n1}, & y_{n1}x_{12} + y_{n2}x_{22} + \dots\dots + y_{nn}x_{n2}, & \dots\dots, & \dots\dots \end{vmatrix} A.$$

These results must be equal since ω_n selects the same terms from a product whatever be the intermediate steps of the work, provided no reductions are made among the units. Thus we have the rule for multiplying determinants. I shall obtain it again in a moment.

Assume a system of functions as follows:

$$\begin{aligned} h_1 &= z_{11}\omega_{n-1}(\alpha_1) + z_{12}\omega_{n-1}(\alpha_2) + \dots\dots + z_{1n}\omega_{n-1}(\alpha_n), \\ h_2 &= z_{21}\omega_{n-1}(\alpha_1) + z_{22}\omega_{n-2}(\alpha_2) + \dots\dots + z_{2n}\omega_{n-1}(\alpha_n), \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ h_n &= z_{n1}\omega_{n-1}(\alpha_1) + z_{n2}\omega_{n-2}(\alpha_2) + \dots\dots + z_{nn}\omega_{\ -1}(\alpha_n), \end{aligned}$$

where $\omega_{n-1}(\alpha_\mu) = \omega_{n-1}\alpha_\mu\alpha_{\mu+1} \dots\dots \alpha_n\alpha_1\alpha_2 \dots\dots \alpha_{\mu-2}$.

Let $\omega_{n-1}^\lambda h_1h_2 \dots\dots h_\lambda$ designate the result obtained by taking out of the product $h_1h_2 \dots\dots h_\lambda$ all those terms containing the product of λ distinct cyclic groups of units such as $l_\mu l_{\mu+1} \dots\dots l_n l_1 l_2 \dots\dots l_{\mu-1}$. ω_{n-1}^λ is distributive over a sum, being merely selective; hence the products obtained may be separately considered. The same reasoning which showed that

$$\omega_\kappa\alpha_1\alpha_2(\omega_\mu\alpha_3\alpha_4\ldots\ldots)\ldots\ldots\alpha_\kappa=\omega_\kappa\alpha_1\alpha_2\ldots\ldots\alpha_\kappa$$

shows that $\omega_{n-1}^\lambda h_1h_2\ldots\ldots(\omega_{n-1}^\delta h_\kappa h_{\kappa+1}\ldots\ldots)\ldots\ldots h_\lambda$,

where ω_{n-1}^δ operates on any δ functions, $\delta<\lambda$.

In the first place,

$$\omega_{n-1}^2[\omega_{n-1}(\alpha_1)\omega_{(n-1)}(\alpha_1)]=0.$$

For

$$\begin{aligned}\omega_{n-1}(\alpha_1)&=C_nl_1l_2\ldots\ldots l_{n-1}+C_1l_2l_3\ldots\ldots l_n+\ldots\ldots\\&=-A[C_nl_n+C_1l_1+C_2l_2+C_3l_3+\ldots\ldots]\end{aligned}$$

Hence

$$\omega_{n-1}^2[\omega_{n-1}(\alpha_1)\omega_{n-1}(\alpha_1)]=A^2[(C_nC_1-C_1C_n)l_nl_1+\ldots\ldots]=0.$$

Also $\omega_{n-1}^2[\omega_{n-1}(\alpha_\lambda)\omega_{n-1}(\alpha_\mu)]=-\omega_{n-1}^2[\omega_{n-1}(\alpha_\mu)\omega_{n-1}(\alpha_\lambda)]$;

for both are linear in the n units. This gives the rule for the signs and shows that no function $\omega_{n-1}(\alpha_\mu)$ can come in twice under ω_{n-1}^λ in the same product. It follows just as in previous cases that

$$\omega_{n-1}^n h_1h_2\ldots\ldots h_n=\begin{vmatrix}z_{11}, & z_{12}, & z_{13}, & \ldots\ldots z_{1n}\\ z_{21}, & z_{22}, & z_{23}, & \ldots\ldots z_{2n}\\ \ldots & \ldots & \ldots & \ldots\ldots\\ z_{n1}, & z_{n2}, & z_{n3}, & \ldots\ldots z_{nn}\end{vmatrix}\omega_{n-1}^n[\omega_{n-1}(\alpha_1)\omega_{n-1}(\alpha_2)\ldots\ldots\omega_{n-1}(\alpha_n)]. \qquad (8)$$

As before shown,

$$\left.\begin{aligned}\omega_{n-1}(\alpha_1)&=-A[C_{1n}l_n+C_{11}l_1+C_{12}l_2+\ldots\ldots+C_{1,\,n-1}l_{n-1}],\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ \omega_{n-1}(\alpha_n)&=-A[C_{nn}l_n+C_{n1}l_1+C_{n2}l_2+\ldots\ldots+C_{n,\,n-1}l_{n-1}],\end{aligned}\right\} \qquad (B)$$

where C_{1n} is the coefficient of $l_1l_2\ldots\ldots l_{n-1}$ in the product $\alpha_1\alpha_2\ldots\ldots\alpha_{n-1}$; but this is plainly

$$\begin{vmatrix}x_{11}, & x_{12}, & x_{13}, & \ldots\ldots, x_{1,\,n-1}\\ x_{21}, & x_{22}, & x_{23}, & \ldots\ldots, x_{2,\,n-1}\\ \ldots & \ldots & \ldots & \ldots\ldots\\ x_{n-1,\,1}, & x_{n-1,\,2}, & x_{n-1,\,3}, & \ldots\ldots, x_{n-1,\,n-1}\end{vmatrix},$$

and C_{11} is formed from this by dropping the left-hand column and adding on the right the next in cyclic order, and so on for the other C's in a perfectly evident manner. That is, C_{1n} is the minor of the determinant

$$\begin{vmatrix}x_{11}, & x_{12}, & x_{13}, & \ldots\ldots, x_{1n}\\ x_{21}, & x_{22}, & x_{23}, & \ldots\ldots, x_{2n}\\ \ldots & \ldots & \ldots & \ldots\ldots\\ x_{n1}, & x_{n2}, & x_{n3}, & \ldots\ldots, x_{nn}\end{vmatrix}=D,$$

complementary to x_{nn}, C_{11} to x_{n1}, C_{12} to x_{n2}, and so on, but plainly the arrangement is such that

$$\begin{aligned} x_{nn}C_{1n} + x_{n1}C_{11} + x_{n2}C_{12} + x_{n3}C_{13} + \dots + x_{n,\,n-1}C_{1,\,n-1} &= D, \\ x_{\mu n}C_{1n} + x_{\mu 1}C_{11} + x_{\mu 2}C_{12} \qquad\qquad + \dots + x_{\mu,\,n-1}C_{1,\,n-1} &= 0. \end{aligned}$$

Also we have plainly in the case of $\omega_{n-1}(\alpha_\lambda)$,

$$\begin{aligned} x_{\lambda-1,\,n}C_{\lambda n} + x_{\lambda-1,\,1}C_{\lambda 1} + x_{\lambda-1,\,2}C_{\lambda 2} + x_{\lambda-1,\,3}C_{\lambda 3} + \dots + x_{\lambda-1,\,n-1}C_{\lambda,\,n-1} &= D, \\ x_{\mu n}C_{\lambda n} + x_{\mu 1}C_{\lambda 1} + x_{\mu 2}C_{\lambda 2} + \dots\dots\dots\dots + x_{\mu,\,n-1}C_{\lambda,\,n-1} &= 0, \end{aligned}$$

μ being different from $\lambda - 1$.

We have also

$$\begin{aligned} x_{nn}C_{1n} + x_{1n}C_{2n} + x_{2n}C_{3n} + \dots + x_{n-1,\,n}C_{nn} &= D, \\ x_{n\mu}C_{1n} + x_{1\mu}C_{2n} + x_{2\mu}C_{3n} + \dots + x_{n-1,\,\mu}C_{nn} &= 0. \end{aligned}$$

All these equations are immediate consequences of the definition of a determinant combined with the equation

$$\omega_n\alpha_1\alpha_2 \dots \alpha_\kappa\alpha_\kappa \dots = 0.$$

Now making

$$\begin{array}{llll} z_{11} = x_{nn}, & z_{12} = x_{1n}, & z_{13} = x_{2n}, & \dots, z_{1n} = x_{n-1,\,n}, \\ z_{21} = x_{n1}, & z_{22} = x_{11}, & z_{23} = x_{21}, & \dots, z_{2n} = x_{n-1,\,1}, \\ z_{31} = x_{n2}, & z_{32} = x_{12}, & z_{33} = x_{22}, & \dots, z_{3n} = x_{n-1,\,2}, \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots\dots \\ z_{\mu 1} = x_{n,\,\mu-1}, & z_{\mu 2} = x_{1,\,\mu-1}, & z_{\mu 3} = x_{2,\,\mu-1} & \dots, z_{\mu n} = x_{n-1,\,\mu-1}, \end{array}$$

there results

$$\begin{aligned} -\frac{1}{A}h_1 = {} & [x_{nn}C_{1n} + x_{1n}C_{2n} + x_{2n}C_{3n} + \dots]\,l_n \\ & + [x_{nn}C_{11} + x_{1n}C_{21} + x_{2n}C_{31} + \dots]\,l_1 + \dots, \end{aligned}$$

or

$$\left.\begin{aligned} h_1 &= -ADl_n, \\ h_2 &= -ADl_1, \\ h_3 &= -ADl_2, \\ &\dots\dots \\ h_n &= -ADl_{n-1}. \end{aligned}\right\} \qquad \text{(C)}$$

It is plain enough now that ω_{n-1}^n does not differ in meaning from ω_n multiplied by $(-A)^n$. From the group (B) we have

$$\omega_{n-1}^n[\omega_{n-1}(\alpha_1)\omega_{n-1}(\alpha_2)\dots\omega_{n-1}(\alpha_n)] = (-A)^n \begin{vmatrix} C_{1n}, & C_{11}, & C_{12}, & C_{13}, \dots & C_{1,\,n-1} \\ C_{2n}, & C_{21}, & C_{22}, & C_{23}, \dots & C_{n,\,n-1} \\ \dots & \dots & \dots & \dots & \dots \\ C_{nn}, & C_{n1}, & C_{n2}, & C_{n3}, \dots & C_{n,\,n-1} \end{vmatrix} l_nl_1l_2 \dots l_{n-1} \quad (9)$$

and from the group (C)

$$\omega_nh_1h_2 \dots h_n = (-A)^nD^nl_nl_1 \dots l_{n-1}.$$

Substituting in equation (8) and replacing the $z_{11}, z_{12} \ldots.$ by their values,

$$(-A)^n D^n . l_n l_1 \ldots . l_{n-1}$$

$$= \begin{vmatrix} x_{nn}, & x_{1n}, & x_{2n}, & \ldots x_{n-1,n} \\ x_{n1}, & x_{11}; & x_{21}, & \ldots x_{n-1,1} \\ \cdots & \cdots & \cdots & \cdots \\ x_{n,n-1}, & x_{1,n-1}, & x_{2,n-1}, & \ldots x_{n-1,n-1} \end{vmatrix} \times \begin{vmatrix} C_{11}, & C_{12}, & C_{13}, & \ldots C_{1n} \\ C_{21}, & C_{22}, & C_{23}, & \ldots C_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ C_{n1}, & C_{n2}, & C_{n3}, & \ldots . C_{nn} \end{vmatrix} (-A)^n l_n l_1 \ldots l_{n-1}.$$

Hence
$$\begin{vmatrix} C_{11}, & C_{12}, & C_{13}, & \ldots . C_{1n} \\ C_{21}, & C_{22}, & C_{23}, & \ldots . C_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ C_{n1}, & C_{n2}, & C_{n3}, & \ldots . C_{nn} \end{vmatrix} = D^{n-1},$$
a well known theorem.

Let it be required to determine $x_1, x_2, \ldots . x_n$ in the equation

$$\rho = x_1 \omega_{n-1}(\alpha_1) + x_2 \omega_{n-1}(\alpha_2) + \ldots . + x_n \omega_{n-1}(\alpha_n).$$

We have

$$S\alpha_n \rho = x_1 S . [\omega_{n-1}(\alpha_1)] \alpha_n + \ldots . + x_\mu S . [\omega_{n-1}(\alpha_\mu)] \alpha_n + \ldots .$$

We have seen that

$$\omega_{n-1}(\alpha_1) = -A [C_{11} l_1 + C_{12} l_2 + \ldots . + C_{1n} l_n],$$

and also
$$\alpha_n = x_{n1} l_1 + x_{n2} l_2 + \ldots . + x_{nn} l_n.$$

Hence

$$S . [\omega_{n-1}(\alpha_1)] \alpha_n = A [x_{n1} C_{11} + x_{n2} C_{12} + \ldots . + x_{nn} C_{1n}] = AD = \omega_n . \alpha_1 \alpha_2 \ldots . \alpha_n .$$

Also

$$S . [\omega_{n-1}(\alpha_\mu)] \alpha_n = A [x_{n1} C_{\mu 1} + x_{n2} C_{\mu 2} + \ldots . + x_{nn} C_{\mu n}] = 0 .$$

Therefore
$$x_1 = \frac{1}{AD} S\alpha_n \rho, \quad x_2 = \frac{1}{AD} S\alpha_1 \rho, \quad x_\mu = \frac{1}{AD} S\alpha_{\mu-1} \rho .$$

These results may be elegantly reached as follows, viz: The product of α_n into $\omega_{n-1}(\alpha_1)$ is made up of products of l_μ into terms which do contain l_μ and terms which do not contain l_μ but contain the other $n-1$ units. The former products are all vectors; the latter are all scalars and when $\mu = 1, 2, \ldots . n$ their aggregate equals $\omega_n : \alpha_1 \alpha_2 \ldots . \alpha_n$ by definition. Hence $S . [\omega_{n-1}(\alpha_\mu)] \alpha_{\mu-1} = \omega_n(\alpha_\mu)$, and $S . [\omega_{n-1}(\alpha_\mu)] \alpha_\lambda = 0$ for $\omega_n . \alpha_\mu \alpha_{\mu+1} \ldots . \alpha_\lambda \ldots . \alpha_\lambda = 0$ as has been shown already. Hence
$$S . \alpha_{\mu-1} \rho = x_\mu \omega_n(\alpha_\mu) = x_\mu AD .$$

Hence

$$\left. \begin{aligned} AD\rho = \omega_{n-1}(\alpha_1) S\alpha_n \rho + \omega_{n-1}(\alpha_2) S\alpha_1 \rho + \omega_{n-1}(\alpha_3) S\alpha_2 \rho + \ldots . \\ + \omega_{n-1}(\alpha_n) . S\alpha_{n-1} \rho \end{aligned} \right\} \quad (10)$$

Let $\beta_1, \beta_2, \beta_3, \ldots . \beta_n$ be any n vectors, then

$$AD\beta_1 = \omega_{n-1}(\alpha_1) . S\alpha_n \beta_1 + \omega_{n-1}(\alpha_2) . S\alpha_1 \beta_1 + \ldots . + \omega_{n-1}(\alpha_n) . S\alpha_{n-1} \beta_1,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$AD\beta_n = \omega_{n-1}(\alpha_1) . S\alpha_n \beta_n + \omega_{n-1}(\alpha_2) . S\alpha_1 \beta_n + \ldots . + \omega_{n-1}(\alpha_n) . S\alpha_{n-1} \beta_n;$$

and applying ω_n,

$$\begin{aligned}A^n D^n \omega_n . \beta_1 \dots \beta_n &= A^n D^n . AD' = A^{n+1} D^n D' \\ &= \omega_{n-1}^n [\omega_{n-1}(\alpha_1) . \omega_{n-1}(\alpha_2) \dots \omega_{n-1}(\alpha_n)]\end{aligned} \begin{vmatrix} S\alpha_1\beta_1, & S\alpha_2\beta_1, & \dots, & S\alpha_n\beta_1 \\ S\alpha_1\beta_2, & S\alpha_2\beta_2, & \dots, & S\alpha_n\beta_2 \\ \dots & \dots & \dots & \dots \\ S\alpha_1\beta_n, & S\alpha_2\beta_n, & \dots, & S\alpha_n\beta_n \end{vmatrix}.$$

But $\omega_{n-1}^n [\omega_{n-1}(\alpha_1) \omega_{n-1}(\alpha_2) \dots \omega_{n-1}(\alpha_n)] = (-1)^n A^{n+1} D^{n-1}$ by eq. (9); and this is $-A^{n+1}D^{n-1}$, since n is odd. Therefore

$$DD' = -\begin{vmatrix} S\alpha_1\beta_1, & S\alpha_2\beta_1, & \dots & S\alpha_n\beta_1 \\ \dots & \dots & \dots & \dots \\ S\alpha_1\beta_n, & S\alpha_2\beta_n, & \dots & S\alpha_n\beta_n \end{vmatrix},$$

which is the rule for multiplication in another form. If the scalars be written out it takes the ordinary form.

Let $\phi\rho$ be a vector function into which ρ enters linearly; there will be terms of the form $\delta S\gamma_1\gamma_2 \dots \gamma_r\rho$ and terms of the form $z\rho$, where z is a scalar. By aid of eq. (10), ρ in the terms of the form $z\rho$ may be expressed in terms of an arbitrary system of n vectors, $\alpha_1, \alpha_2, \dots \alpha_n$, provided D does not vanish. The vectors δ may be expressed in terms of the same system and $\phi\rho$ will take the form

$$\begin{aligned}\phi\rho = \omega_{n-1}(\alpha_1)[\Sigma S . \gamma_{11}\gamma_{12} \dots \gamma_{1\mu}\rho] + \omega_{n-1}(\alpha_2)[\Sigma S . \gamma_{21}\gamma_{22} \dots \gamma_{2\nu}\rho] + \dots \\ + \omega_{n-1}(\alpha_n)[\Sigma S . \gamma_{n1}\gamma_{n2} \dots \gamma_{n\tau}\rho].\end{aligned}$$

But $\Sigma S . \gamma_1\gamma_2 \dots \gamma_\mu\rho = S . [\Sigma\gamma_1\gamma_2 \dots \gamma_\mu]\rho = S\beta_1\rho$. Hence we have finally

$$\phi\rho = \omega_{n-1}(\alpha_1) . S\beta_1\rho + \omega_{n-1}(\alpha_2) . S\beta_2\rho + \dots + \omega_{n-1}(\alpha_n) . S\beta_n\rho = \gamma. \quad (11)$$

Eq. 11 being given, let it be required to express $\phi^{-1}\gamma = \rho$ in terms of the known operator ϕ. For brevity, write $\phi\rho$ as follows, viz:

$$\phi\rho = \delta_1 S\beta_1\rho + \delta_2 S\beta_2\rho + \dots + \delta_n S\beta_n\rho,$$

and take $$\phi'\rho_1 = \beta_1 S\delta_1\rho_1 + \beta_2 S\delta_2\rho_1 + \dots + \beta_n S\delta_n\rho_1.$$

Then, ρ_1 being any vector whatever,

$$S\rho_1\phi\rho = S\rho\phi'\rho_1, \text{ evidently.}$$

Following Hamilton, on whose work all this is based, the functions ϕ and ϕ' may be called conjugate to each other. Now take $n-1$ vectors, $\lambda_1, \lambda_2, \dots \lambda_{n-1}$ such that $\omega_{n-1} . \lambda_1\lambda_2 \dots \lambda_{n-1} = \gamma = \phi\rho$, it is clear that

$$\begin{aligned}S\lambda_1\phi\rho &= S . \lambda_1\omega_{n-1} . \lambda_1\lambda_2 \dots \lambda_{n-1} &&= \omega_n . \lambda_1\lambda_1\lambda_2 \dots \lambda_{n-1} &&= 0, \\ S\lambda_\kappa\phi\rho &= S . \lambda_\kappa\omega_{n-1} . \lambda_1\lambda_2 \dots \lambda_\kappa \dots \lambda_{n-1} &&= \omega_n . \lambda_\kappa\lambda_1\lambda_2 \dots \lambda_\kappa \dots \lambda_{n-1} &&= 0,\end{aligned}$$

where $\mu = 1, 2, \dots . n-1$. But

$$S\lambda_\kappa\phi\rho = S\rho\phi'\lambda_\kappa = 0,$$

giving $n-1$ conditions which will all be fulfilled if

$$m\rho = \omega_{n-1}.\phi'\lambda_1\phi'\lambda_2 \dots . \phi'\lambda_{n-1},$$

m being a scalar whose value is to be found.

But since $\phi\rho = \omega_{n-1}.\lambda_1\lambda_2 \dots . \lambda_{n-1}$,

$$\rho = \phi^{-1}[\omega_{n-1}.\lambda_1\lambda_2 \dots . \lambda_{n-1}];$$

hence,

$$m\phi^{-1}[\omega_{n-1}.\lambda_1\lambda_2 \dots . \lambda_{n-1}] = \omega_{n-1}.\phi'\lambda_1\phi'\lambda_2 \dots . \phi'\lambda_{n-1}. \tag{12}$$

Let λ_n be a vector such that $\omega_n.\lambda_1\lambda_2 \dots . \lambda_n \neq 0$; then

$$\begin{aligned} S.\phi'\lambda_n\phi^{-1}[\omega_{n-1}.\lambda_1\lambda_2 \dots . \lambda_{n-1}] &= S.\lambda_n\phi\phi^{-1}[\omega_{n-1}.\lambda_1\lambda_2 \dots . \lambda_{n-1}] \\ = S.\lambda_n[\omega_{n-1}.\lambda_1 \dots . \lambda_{n-1}] &= S.[\omega_{n-1}.\lambda_1\lambda_2 \dots . \lambda_{n-1}]\,\lambda_n = \omega_n.\lambda_1\lambda_2 \dots . \lambda_n. \end{aligned}$$

Also

$$\begin{aligned} S.\phi'\lambda_n[\omega_{n-1}.\phi'\lambda_1\phi'\lambda_2 \dots . \phi'\lambda_{n-1}] &= S.[\omega_{n-1}.\phi'\lambda_1\phi'\lambda_2 \dots . \phi'\lambda_{n-1}]\,\phi'\lambda_n \\ &= \omega_n.\phi'\lambda_1\phi'\lambda_2 \dots . \phi'\lambda_n. \end{aligned}$$

Therefore, multiplying eq. (12) by $\phi'\lambda_n$ and taking scalars, m is given by the equation

$$m = \frac{\omega_n.\phi'\lambda_1\phi'\lambda_2 \dots . \phi'\lambda_n}{\omega_n.\lambda_1\lambda_2 \dots . \lambda_n}.$$

As m is a homogeneous function of degree 0 in $\lambda_1, \lambda_2, \dots . \lambda_n$, it is not altered by linear transformation; its value is therefore independent of the particular auxiliary system chosen in any given case. This is easy to see, viz: taking

$$\begin{gathered} \lambda_1 = c_{11}l_1 + c_{12}l_2 + \dots . + c_{1n}l_n, \\ \dots\dots\dots\dots\dots\dots \\ \lambda_\mu = c_{\mu 1}l_1 + c_{\mu 2}l_2 + \dots . + c_{\mu n}l_n, \end{gathered}$$

we have

$$\omega_n.\lambda_1 \dots . \lambda_n = \begin{vmatrix} c_{11}, & c_{12}, & \dots . & c_{1n} \\ \cdot & \cdot & \cdot & \cdot \\ c_{n1}, & c_{n2}, & \dots . & c_{nn} \end{vmatrix} A = CA.$$

Also,

$$\begin{gathered} \phi'\lambda_1 = c_{11}\phi'l_1 + c_{12}\phi'l_2 + \dots . + c_{1n}\phi'l_n, \\ \dots\dots\dots\dots\dots\dots \\ \phi'\lambda_\mu = c_{\mu 1}\phi'l_1 + c_{\mu 2}\phi'l_2 + \dots . + c_{nn}\phi'l_n, \end{gathered}$$

and

$$\omega_n.\phi'\lambda_1\phi'\lambda_2 \dots . \phi'\lambda_n = C\omega_n.\phi'l_1\phi'l_2 \dots . \phi'l_n.$$

Hence

$$m = \frac{\omega_n.\phi'\lambda_1\phi'\lambda_2 \dots . \phi'\lambda_n}{\omega_n.\lambda_1\lambda_2 \dots . \lambda_n} = \frac{1}{A}\,\omega_n.\phi'l_1\phi'l_2 \dots . \phi'l_n.$$

Making $\phi = \phi + g$ and therefore $\phi' = \phi' + g$, and repeating the calculation of m for the new function, it comes out

$$mg = \frac{\omega_n . (\phi' + g)\lambda_1 (\phi' + g)\lambda_2 \dots . (\phi' + g)\lambda_n}{\omega_n . \lambda_1\lambda_2 \dots . \lambda_n}$$
$$= \frac{1}{A}\, \omega_n . (\phi' + g)\, l_1 (\phi' + g)\, l_2 \dots . (\phi' + g)\, l_n .$$

That is,

$$mg = m + m_1 g + m_2 g^2 + \dots . + m_\mu g^\mu + \dots . + m_n g^n,$$

the values of $m_1, m_2 \dots .$ being evident, and in particular, $m_n = 1$. From eq. (12),

$$\begin{aligned} & mg\,(\phi + g)^{-1}[\omega_{n-1} . \lambda_1\lambda_2 \dots . \lambda_{n-1}] \\ & = \omega_{n-1} . [(\phi' + g)\lambda_1 (\phi' + g)\lambda_2 \dots . (\phi' + g)\lambda_{n-1}] \\ & = \omega_{n-1} . \phi'\lambda_1\phi'\lambda_2 \dots . \phi'\lambda_{n-1} + g[A] + g^2[B] + \dots . + g^{n-1}\omega_{n-1} . \lambda_1\lambda_2 \dots . \lambda_{n-1} \\ & = m\phi^{-1}[\omega_{n-1} . \lambda_1\lambda_2 \dots \lambda_{n-1}] + g[A] + g^2[B] + \dots . + g^{n-1}\omega_{n-1} . \lambda_1\lambda_2 \dots . \lambda_{n-1} . \end{aligned}$$

Operating on both sides with $(\phi + g)$,

$$\begin{aligned} & [m + m_1 g + m_2 g^2 + \dots . + m_\mu g^\mu + \dots . m_n g^n]\,\omega_{n-1} . \lambda_1\lambda_2 \dots . \lambda_{n-1} \\ & = mg\omega_{n-1} . \lambda_1\lambda_2 \dots . \lambda_{n-1} = m\omega_{n-1} . \lambda_1\lambda_2 \dots . \lambda_{n-1} + g\,\{\phi[A] \\ & \quad + m\phi^{-1}[\omega_{n-1} . \lambda_1\lambda_2 \dots . \lambda_{n-1}]\} + g^2\{\phi[B] + [A]\} \\ & \quad + g^3\{\phi[C] + [B]\} + \dots . + g^{n-2}\{\phi[M] + [L]\} \\ & \quad + g^{n-1}\{\phi\,[\omega_{n-1} . \lambda_1 \dots . \lambda_{n-1}] + [M]\} + g^n\omega_{n-1} . \lambda_1\lambda_2 \dots . \lambda_{n-1} . \end{aligned}$$

Now g being any scalar whatever, its coefficients on both sides are equal, and this is assured by noting that the coefficients of g^ν and g^n are identical, since $m_n = 1$. Hence

$$\phi[A] + m\phi^{-1}\omega_{n-1}(\lambda_1) = m_1\omega_{n-1}(\lambda_1),$$

or

$$[A] = m_1\phi^{-1}\omega_{n-1}(\lambda_1) - m\phi^{-2}\omega_{n-1}(\lambda_1),$$

that is, $[A]$ is a linear and vector function of $\omega_{n-1} . (\lambda_1)$, and the same may be shown for $[B]$, ... But we have

$$\begin{aligned} \phi\omega_{n-1}(\lambda) + [M] \quad & = m_{n-1}\omega_{n-1}(\lambda_1), \\ \phi[M] + [L] \quad & = m_{n-2}\omega_{n-1}(\lambda_1), \\ \phi[L] + [K] \quad & = m_{n-3}\omega_{n-1}(\lambda_1), \\ \dots\dots\dots\dots & \dots\dots\dots\dots \\ \phi[C] + [B] \quad & = m_3\omega_{n-1}(\lambda_1), \\ \phi[B] + [A] \quad & = m_2\omega_{n-1}(\lambda_1), \\ \phi[A] + m\phi^{-1}\omega_{n-1}(\lambda_1) & = m_1\omega_{n-1}(\lambda_1). \end{aligned}$$

Hence

$$\begin{aligned}
[M] &= [m_{n-1} - \phi]\,\omega_{n-1}(\lambda_1),\\
[L] &= [m_{n-2} - m_{n-1}\phi + \phi^2]\,\omega_{n-1}(\lambda_1),\\
[K] &= [m_{n-3} - m_{n-2}\phi + m_{n-1}\phi^2 - \phi^3]\,\omega_{n-1}(\lambda_1),\\
&\dots\dots\dots\dots\dots\dots\dots\dots\\
[A] &= [m_2 - m_3\phi + m_4\phi^2 - m_5\phi^3 + \dots + m_{n-1}\phi^{n-3} - \phi^{n-2}]\,\omega_{n-1}(\lambda_1),
\end{aligned}$$

$$m\phi^{-1}\omega_{n-1}(\lambda_1) = [m_1 - m_2\phi + m_3\phi^2 - \dots - m_{n-1}\phi^{n-2} + \phi^{n-1}]\,\omega_{n-1}(\lambda_1).$$

$\omega_{n-1}(\lambda_1)$ is an arbitrary vector; hence if it be omitted there will remain a symbolical equation in ϕ true for any vector that may be inserted, therefore

$$m\phi^{-1} = m_1 - m_2\phi + m_3\phi^2 - \dots - m_{n-1}\phi^{n-2} + \phi^{n-1}, \tag{12}$$

or $$\phi^n - m_{n-1}\phi^{n-1} + m_{n-2}\phi^{n-2} - \dots + m_3\phi^3 - m_2\phi^2 + m_1\phi - m = 0. \tag{13}$$

The coefficients $m_1, m_2, \dots m_{n-1}$ are all independent of the vectors $\lambda_1 \dots \lambda_n$, for they can be expressed in terms of the units $l_1 \dots l_n$ and the constant vectors of ϕ'.

Equation (13) may be written

$$(\phi + g_1)(\phi + g_2)\dots(\phi + g_n) = 0,$$

which means that n vectors $\delta_1, \delta_2, \dots \delta_n$ can be found such that each of them satisfies a relation of the form

$$(\phi + g_\kappa)\,\delta_\kappa = 0, \text{ or } \phi\delta_\kappa = -g_\kappa\delta_\kappa.$$

But plainly $g_1, g_2, \dots g_n$ are the roots of $m_g = 0$. Of course there is a relation of the same kind for ϕ', viz:

$$(\phi' + h_1)(\phi' + h_2)\dots(\phi' + h_n) = 0,$$

where $h_1 \dots h_n$ are the roots of an equation

$$m_h = 0,$$

m_h being the quantity related to ϕ' as m_g is to ϕ. The operation $(\phi + g)^{-1}$ has already been determined as follows, viz:

$$m_g(\phi + g)^{-1}[\omega_{n-1}.\lambda_1 \dots \lambda_{n-1}] = \omega_{n-1}.[(\phi'+g)\lambda_1(\phi'+g)\lambda_2 \dots (\phi'+g)\lambda_{n-1}]$$

and for $(\phi' + h)^{-1}$ we have therefore

$$m_h(\phi'+h)^{-1}[\omega_{n-1}.\lambda_1 \dots \lambda_{n-1}] = \omega_{n-1}.[(\phi+h)\lambda_1(\phi+h)\lambda_2 \dots (\phi+h)\lambda_{n-1}],$$

where $$m_h = \frac{1}{A}\,\omega_n.(\phi + h)l_1(\phi + h)l_2 \dots (\phi + h)l_n.$$

Let

$$D^n\rho + P_1D^{n-1}\rho + P_2D^{n-2}\rho + \dots + P_n\rho = 0, \text{ where } D^\mu = \frac{d^\mu}{dt^\mu}, \tag{18}$$

be a linear differential equation of the n^{th} order in ρ with uniform coefficients, and t be the independent scalar variable; then, if

$$\rho = x_1\alpha_1 + x_2\alpha_2 + \dots + x_n\alpha_n$$

be an integral of (18), $x_1, x_2, \dots x_n$ are a system of integrals of the scalar equation

$$D^n x + P_1 D^{n-1} x + \dots + P_n x = 0, \tag{19}$$

provided $\omega_n . \alpha_1\alpha_2 \dots \alpha_n$ does not vanish. For substituting the value of ρ in eq. (18), the coefficients of $\alpha_1, \alpha_2, \dots \alpha_n$ must separately vanish; but these coefficients are the results of substituting $x_1, x_2, \dots x_n$ successively in eq. (19). Evidently, also, if $x_1, x_2, \dots x_n$ are a system of independent integrals of eq. (19), the equation in ρ will be satisfied by

$$\rho = x_1\alpha_1 + x_2\alpha_2 + \dots + x_n\alpha_n,$$

$\alpha_1, \dots \alpha_n$ being any constant vectors whatever.

If ρ be expressed in terms of any other set of independent vectors as $\beta_1, \beta_2, \dots \beta_n$, the scalar coefficients will still be integrals of (19). For we have, changing the notation of eq. (10),

$$\alpha_1 = c_{11}\beta_1 + c_{12}\beta_2 + \dots + c_{1n}\beta_n,$$
$$\dots\dots\dots\dots\dots\dots\dots$$
$$\alpha_n = c_{n1}\beta_1 + c_{n2}\beta_2 + \dots + c_{nn}\beta_n,$$

whence

$$\rho = (c_{11}x_1 + c_{21}x_2 + \dots + c_{n1}x_n)\beta_1 + \dots + (c_{1n}x_1 + c_{2n}x_2 + \dots + c_{nn}x_n)\beta_n.$$

If ρ be an integral of eq. (18), $\phi\rho$ will also be an integral. For, evidently,

$$\phi\,[D^n\rho + P_1 D^{n-1}\rho + \dots + P_n\rho] = 0.$$

But ϕ is commutative with D as is well known, hence

$$D^n\phi\rho + P_1 D^{n-1}\phi\rho + \dots + P_n D\rho = 0.$$

This is also clear from the fact that $\phi\rho$ is obtained from ρ by a linear substitution. For, taking

$$\rho = x_1\alpha_1 + x_2\alpha_2 + \dots + x_n\alpha_n,$$

and $$\phi\rho = \alpha_1 S\beta_1\rho + \alpha_2 S\beta_2\rho + \dots + \alpha_n S\beta_n\rho,$$

we have $$\phi\alpha_1 = \alpha_1 S\beta_1\alpha_1 + \alpha_2 S\beta_2\alpha_1 + \dots + \alpha_n S\beta_n\alpha_1,$$

or $$\phi\alpha_1 = a_{11}\alpha_1 + a_{12}\alpha_2 + \dots + a_{1n}\alpha_n,$$

and in the same way,

$$\phi\alpha\mu = a_{\mu 1}\alpha_1 + a_{\mu 2}\alpha_2 + \dots + a_{\mu n}\alpha_n.$$

But $$\phi\rho = x_1\phi\alpha_1 + x_2\phi\alpha_2 + \dots + x_n\phi\alpha_n;$$

hence

$$\phi\rho = (a_{11}x_1 + a_{21}x_2 + \dots + a_{n1}x_n)\alpha_1 + (a_{12}x_1 + a_{22}x_2 + \dots + a_{n2}x_n)\alpha_2 + \dots + (a_{1n}x_1 + a_{2n}x_2 + \dots + a_{nn}x_n)\alpha_n. \tag{20}$$

So that ρ may be transformed by changing the vectors of reference or by applying ϕ, and the coefficients will remain integrals of eq. (19). If

$$\rho = x_1\alpha_1 + x_2\alpha_2 + \ldots . + x_n\alpha_n$$

is an integral of eq. (18), it follows that $S\delta\rho = x_1 S\delta\alpha_1 + x_2 S\delta\alpha_2 + \ldots . + x_n S\delta\alpha_n$ is a general integral of eq. (19); and the same is true of $S.\delta\phi\rho$. Now let it be required to determine δ in such a manner that $S.\delta\phi\rho = hS\delta\rho$, h being a constant scalar; that is, to determine an integral of eq. (19) which is unaltered by a certain linear transformation, except as to a constant factor. We have

$$S.\delta\phi\rho = hS\delta\rho; \text{ but } S.\delta\phi\rho = S\rho\phi'\delta,$$

hence $$hS\delta\rho = S\rho\phi'\delta, \text{ or } S.\rho(h\delta - \phi'\delta) = 0$$

whatever be ρ; but this requires that $\phi'\delta = h\delta$, hence h must be a root of the equation in ϕ' corresponding to the equation (13) in ϕ, and δ must be the corresponding vector. There are, therefore, n integrals of eq. (19) which are only multiplied by a constant by the transformation ϕ. If

$$S.\delta\phi\rho = hS\delta\rho, \text{ then } S.\delta\phi^2\rho = h^2 S\delta\rho.$$

For $$S.\delta\phi^2\rho = S.\phi'\delta\phi\rho = S.\phi'^2\delta\rho = S.\rho\phi'^2\delta = S.\rho h^2\delta = h^2 S\rho\delta.$$

Hence the integrals $S\delta_1\rho$, $S\delta_2\rho$, ... $S\delta_n\rho$ determined by the above process are independent, as may be seen in the usual way.

The above simple process may be varied as follows, viz: We have

$$\begin{aligned} S.\delta\phi\rho &= (a_{11}x_1 + a_{12}x_2 + \ldots . + a_{1n}x_n)\, S\delta\alpha_1 + \ldots . + (a_{1n}x_1 + a_{2n}x_2 + \ldots . a_{nn}x_n)\, S\delta\alpha_n \\ &= hS\delta\rho = h(x_1 S\delta\alpha_1 + x_2 S\delta\alpha_2 + \ldots . + x_n S\delta\alpha_n); \end{aligned}$$

giving n equations in h, viz:

$$\begin{aligned} (a_{11} - h)\, S\delta\alpha_1 + a_{12} S\delta\alpha_2 \quad &+ \ldots . + a_{1n} S\delta\alpha_n = 0, \\ a_{21} S\delta\alpha_1 + (a_{22} - h)\, S\delta\alpha_2 &+ \ldots . + a_{2n} S\delta\alpha_n = 0, \\ \ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots \\ a_{n1} S\delta\alpha_1 + a_{n2} S\delta\alpha_2 \quad &+ \ldots . + a_{nn} S\delta\alpha_n = 0; \end{aligned}$$

whence h must be a root of the equation

$$\begin{vmatrix} a_{11} - h, & a_{12} & , a_{13}, \ldots ., & a_{1n} \\ a_{21} & , a_{22} - h, & a_{23}, \ldots ., & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & , a_{n2} & , a_{n3}, \ldots ., & a_{nn} - h \end{vmatrix} = 0. \qquad (21)$$

If h_1 be a root of eq. (21), it may be substituted in the preceding equations, and the result is n conditions to determine δ_1; hence the statement that δ_1, δ_2, ... δ_n are determinate vectors is fully justified.

We know that the roots of eq. (21) are identical with those of

$$m_h = 0, \tag{22}$$

therefore the coefficients of the successive powers of h in eq. (21) are invariants. This fact is otherwise sufficiently clear, for those coefficients depend only upon certain constant vectors and not at all upon the variables in ρ.

In case the eq. $\quad m_h = 0$

has a pair of equal roots, the properties of the corresponding integrals may be obtained with great ease, viz: It is clear that δ_1 may be so taken as to satisfy the relation $\quad S\delta_1\phi\rho = S\rho\phi'\delta_1 = h_1 S\rho\delta_1,$

—h_1 being one of the equal roots. Then taking γ_2 so as to satisfy the equation

$$\phi'\gamma_2 = \delta_1 + h_1\gamma_2, \tag{23}$$

we have $\quad S\rho\phi'\gamma_2 = S\gamma_2\phi\rho = S\delta_1\rho + h_1 S\gamma_2\rho.$

If more than two roots of eq. (22) are equal, let their number be λ and let the vectors $\gamma_2, \gamma_3, \ldots \gamma_\lambda$ be given by the following equations, viz:

$$\begin{aligned}
&\phi'\gamma_2 = \delta_1 + h_1\gamma_2, \\
&\phi'\gamma_3 = \delta_1 + \gamma_2 + h_1\gamma_3, \\
&\phi'\gamma_4 = \delta_1 + \gamma_2 + \gamma_3 + h_1\gamma_4, \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
&\phi'\gamma_\lambda = \delta_1 + \gamma_2 + \gamma_3 + \ldots + \gamma_{\lambda-1} + h_1\gamma_\lambda.
\end{aligned}$$

Then the integrals and their forms after the substitution ϕ will be

$$\left.\begin{aligned}
&S\delta_1\rho\,;\ S\delta_1\phi\rho = h_1 S\delta_1\rho, \\
&S\gamma_2\rho\,;\ S\gamma_2\phi\rho = S\delta_1\rho + h_1 S\gamma_2\rho, \\
&S\gamma_3\rho\,;\ S\gamma_3\phi\rho = S\delta_1\rho + S\gamma_2\rho + h_1 S\gamma_3\rho, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&S\gamma_\lambda\rho\,;\ S\gamma_\lambda\phi\rho = S\delta_1\rho + S\gamma_2\rho + S\gamma_3\rho + \ldots + h_1 S\gamma_\lambda\rho\,;
\end{aligned}\right\} \tag{24}$$

all these are obtained by multiplying by ρ and taking the scalars, remembering that $\quad S\rho\phi'\gamma = S\gamma\phi\rho.$

It may be noted that γ_2 cannot be parallel to δ_1; for, making $\gamma_2 = z\delta_1$ and substituting in eq. (23), we find

$$z\phi'\delta_1 = \delta_1 + h_1 z\delta_1,$$

or $\quad h_1 z\delta_1 = \delta_1 + h_1 z\delta_1, \text{ or } \delta_1 = 0,$

which is not in general true.

We may obtain the integrals in a much simpler form by taking

$$\begin{aligned} \phi'\gamma_2 &= h_1(\gamma_2+\delta_1), \\ \phi'\gamma_3 &= h_1(\gamma_3+\gamma_2), \\ &\ldots\ldots\ldots\ldots \\ \phi'\gamma_\lambda &= h_1(\gamma_\lambda+\gamma_{\lambda-1}), \end{aligned}$$

and the integrals before and after the substitution ϕ will be as follows, viz:

$$\left.\begin{aligned} &S\delta_1\rho\,;\ S\delta_1\phi\rho = h_1 S\delta_1\rho, \\ &S\gamma_2\rho;\ S\gamma_2\phi\rho = h_1[S\gamma_2\rho + S\delta_1\rho], \\ &S\gamma_3\rho;\ S\gamma_3\phi\rho = h_1[S\gamma_3\rho + S\gamma_2\rho], \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &S\gamma_\lambda\rho\,;\ S\gamma_\lambda\phi\rho = h_1[S\gamma_\lambda\rho + S\gamma_{\lambda-1}\rho]. \end{aligned}\right\} \tag{25}$$

The form of the system (25) is that obtained by Jordan (Cours d'Analyse, III, No. 144) by a somewhat complex analysis.

It remains to prove that the integrals of the system (25), together with these determined by the remaining vectors $\delta_{\lambda+1}\ldots\ldots\delta_n$ which satisfy the equation

$$\phi'\delta_\kappa - h_\kappa\delta_\kappa = 0,$$

form an independent system. If they satisfy any linear relation, let it be the following:

$$c_1S\delta_1\rho + c_2S\gamma_2\rho + \ldots\ldots + c_\lambda S\gamma_\lambda\rho + c_{\lambda+1}S\delta_{\lambda+1}\rho + \ldots\ldots + c_nS\delta_n\rho = 0,$$

which may be written

$$S\rho\,.\,[c_1\delta_1 + c_2\gamma_2 + \ldots\ldots + c_\lambda\gamma_\lambda + c_{\lambda+1}\delta_{\lambda+1} + \ldots\ldots + c_n\delta_n] = 0 = S\rho\,[\eta], \tag{26}$$

$[\eta]$ representing the bracketed vectors. Eq. (26) is satisfied if ρ is perpendicular to $[\eta]$, which may happen only for certain directions; but, as t varies, ρ turns in an infinite number of directions; and it follows that eq. (26) cannot be true for all values of ρ unless $[\eta]=0$. We must have then

$$c_1\delta_1 + c_2\gamma_2 + \ldots\ldots + c_\lambda\gamma_\lambda + c_{\lambda+1}\delta_{\lambda+1} + \ldots\ldots + c_n\delta_n = 0. \tag{27}$$

Operating on eq. (27) with

$$(\phi'-h_1)(\phi'-h_{\lambda+1})(\phi'-h_{\lambda+2})\ldots\ldots(\phi'-h_n),$$

the vectors δ_1, $\delta_{\lambda+1}$, ... δ_n all disappear, leaving the equation

$$(\phi'-h_1)(\phi'-h_{\lambda+1})\ldots\ldots(\phi'-h_n)[c_2\gamma_2 + \ldots\ldots + c_\lambda\gamma_\lambda] = 0. \tag{28}$$

Eq. (26) may be satisfied in two ways: (1) if the vectors $\gamma_2\ldots\ldots\gamma_n$ are parallel respectively to any vectors of the system δ_1, $\delta_{\lambda+1}$, ... δ_n; or (2) if $c_2\gamma_2 + \ldots\ldots + c_\lambda\gamma_\lambda = 0$. But, referring back to the assumed values of $\gamma_2, \ldots\ldots \gamma_\lambda$, we have

$$(\phi'-h_1)\,\gamma_2 = \delta_1, \tag{29}$$

and if $\gamma_2 - z\delta_1$, the first member vanishes leaving $\delta_1 = 0$, which is false. If again $\gamma_2 = z\delta_\kappa$, we have

$$z\phi'\delta_\kappa - zh_1\delta_\kappa = \delta_1$$

or

$$z(h_\kappa - h_1)\,\delta_\kappa = \delta_1;$$

but δ_κ cannot be parallel to δ_1 unless the roots h_κ and h_1 of eq. (22) are equal, as may be plainly seen from the remark immediately following eq. (21); and these roots are, by hypothesis, not equal. Hence γ_2 cannot be parallel to any of the vectors $\delta_1, \delta_{\lambda+1}, \ldots \delta_n$. Moreover, we have

$$\phi'\gamma_\lambda - h_1\gamma_\lambda = h_1\gamma_{\lambda-1}. \qquad (30)$$

Making $\gamma_\lambda = z\delta_\kappa$ in eq. (30), it becomes

$$z(h_\kappa - h_1)\,\delta_\kappa = h_1\gamma_{\lambda-1},$$

that is, if any vector γ_λ is parallel to δ_κ, the next preceding it is also parallel to δ_κ. This leads directly back to the assertion that γ_2 is parallel to δ_κ, which we know to be false. Therefore, if eq. (29) is true we must have

$$c_2\gamma_2 + \ldots + c_\lambda\gamma_\lambda = 0.$$

This reduces eq. (27) to the following, viz:

$$c_1\delta_1 + c_{\lambda+1}\delta_{\lambda+1} + \ldots + c_n\delta_n = 0. \qquad (31)$$

Operating on eq. (31) with ϕ' successively $n - \lambda$ times, we have

$$\begin{aligned}
c_1\delta_1 + c_{\lambda+1}\delta_{\lambda+1} \quad &+ \ldots + c_n\delta_n \quad = 0,\\
h_1c_1\delta_1 + h_{\lambda+1}c_{\lambda+1}\delta_{\lambda+1} &+ \ldots + h_nc_n\delta_n \quad = 0,\\
\ldots\ldots\ldots\ldots\ldots&\ldots\ldots\ldots\ldots\ldots\\
h_1^{n-\lambda}c_1\delta_1 + h_{\lambda+1}^{n-\lambda}c_{\lambda+1}\delta_{\lambda+1} &+ \ldots + h_n^{n-\lambda}c_n\delta_n = 0.
\end{aligned}$$

Eliminating $c_1\delta_1, c_{\lambda+1}\delta_{\lambda+1}, \ldots c_n\delta_n$, it is found that the determinant

$$\begin{vmatrix}
1 \;, & 1 \;, & \ldots, & 1\\
h_1 \;, & h_{\lambda+1}, & \ldots, & h_n\\
h_1^2 \;, & h_{\lambda+1}^2, & \ldots, & h_n^2\\
\ldots & \ldots & \ldots & \ldots\\
h_1^{n-\lambda}, & h_{\lambda+1}^{n-\lambda}, & \ldots, & h_n^{n-\lambda}
\end{vmatrix}$$

must vanish. But the value of this determinant is the product of all the differences $(h_{\lambda+1} - h_1) \ldots$, etc., and cannot vanish because no two of these roots are equal. Therefore there can be no linear relation among the integrals as determined above.

The operator ϕ represents the substitution which takes place among the independent integrals when t goes around a critical point and its constants necessarily change in passing from one critical point to another.

A Problem suggested in the Geometry of Nets of Curves and applied to the Theory of Six Points having multiply Perspective Relations.

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1. A 1.1 correspondence between the curves of two nets of curves, one in a plane Ξ, the other in a plane Π, may be established by making four curves of one net no three of which are linearly related, correspond to four such curves of the other net. Let two arbitrary curves of the net in $\genfrac{}{}{0pt}{}{\Xi}{\Pi}$ intersect, besides in the common base point system (if any), in a group of $\genfrac{}{}{0pt}{}{p}{q}$ points; the 1.1 correspondence between the curves of the nets establishes a $p.q$ correspondence between the points of the two planes Ξ, Π.

In particular, for $p = q = 1$, there is a 1.1 correspondence between the points of the planes, *i. e.* a Cremona transformation changes one plane of points into the other. The geometry of the net of curves and the points* of Ξ is transformed into a similar geometry of the net of curves and the points of Π. One and so simultaneously both of the nets of curves may be set in correspondence with the net of lines in a plane Ω, and hence also arises a Cremona transformation between the points of Ω and Ξ, and of Ω and Π. Thus it is natural to speak of the geometry of the net of curves and the points of Ξ or Π in the terms adopted in the (ordinary) geometry of the lines and points of a plane Ω.

In the plane Ω let a figure E be determined by certain points; these determining points of E are subject to certain conditions or limitations as to generality of position in order that the figure E may enjoy certain properties. Assuming as fixed the correspondence between the curves of Π and the lines of Ω, the corresponding figure F of Π is determined by the corresponding points, which are subject to the corresponding conditions (which now involve the determining points of F separately and these points together with the base point system of the net in Π) in order that the figure F may enjoy the corresponding properties.

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* That is, geometric constructions and theorems which are or *may be* completely expressed in terms of points and curves of the net.

However, after the general character of the net in Π, that is, the order of the curves and the full nature of the base point, or say *principal*, system has been determined upon, this principal system depends still upon certain arbitrary constants or parameters connected with the determination of position in Π. Now considering these parameters of the principal system of Π at our disposal, it will in general be possible, by the suitable determination of an equal number of parameters, to relieve the determining points of F from all the conditions which involve also the principal system; this leaves the points of F subject to say *internal* conditions only.

Then, when the points of F have been fixed subject to the remaining internal conditions, a principal system (definite or with certain parameters still remaining equal in number to the excess of the original number of parameters over the number of conditions from which the points of F have been relieved) may be found; that is, a net of curves in Π of the required nature may be found, so that in the geometry of that net of curves the points of F shall determine a figure F having the required properties corresponding to the properties of the figure E in the plane Ω.

It is possible that the points of F, in particular positions, satisfy not only the required internal conditions, but also others likewise internal, and such as to give to the principal system an equal number of parameters. (See §3.)

Especially noteworthy (*e. g.*, § 3, end, §4 fg.) are the cases in which in this way the points of F are released entirely from conditions. This will be possible only when the number of conditions imposed upon them for a fixed principal system in Π is equal to or less than the number of parameters of the principal system. (The converse is not true; see §3, the principal system has six parameters, for $n = 7$ there are only five conditions, but of these two are strictly *internal* and cannot be relieved by suitable choice of principal system.)

2. In the following paper I wish, when the net of curves under consideration in Π is the net of conics through three points, to make this determination of the principal system for two figures F; in the second case the figure will receive further consideration on its own account.

Consider between the points of the two planes Ω, Π a quadratic transformation with as principal system in each plane three distinct non-collinear points, say in Π, $H_1H_2H_3$. Then with well-understood exceptions relating to the principal system in Ω, in the plane Π to a point in Ω corresponds a point, to a line in Ω, a conic passing through the three points H_ι, and to the net of lines in Ω, the net of conics through the three points H_ι. The (ordinary) geometry of the points and lines of Ω is transformed into the geometry of the points and this net of conics in the plane Π, say for brevity into the geometry of the plane $\Pi_{C^2(H_1H_2H_3)}$, where the subscript indicates the net of curves in Π corresponding to the net of lines in Ω; or, where no ambiguity will arise, the geometry of the plane $\Pi_{H_1H_2H_3}$ (indicating merely the principal points of the net) or even of the plane Π_H. The principal system consisting of three points has six parameters.

Case I. 3. The simplest possible case will be taken as a good illustration.

In Ω let the figure E consist of n points $(n > 2)$ subject to the $n - 2$ conditions that they shall lie in the same straight line.

In Π_H likewise the n points of F will be subject to the $n - 2$ conditions that they shall all lie on a conic passing through $H_1H_2H_3$.

First, $n > 5$. Then these $n - 2$ conditions are equivalent to the $n - 5$ *internal* conditions that they shall all lie on the conic determined by any five of them. and the three conditions that this conic shall pass through H_1, H_2, and H_3. Let n points $P_\kappa\,(\kappa=1\ldots n)$ be subject merely to the $n - 5$ internal conditions that they shall be conconical* on say conic C^2. We expect to find a principal system H with $(6 - 3 =)\, 3$ remaining parameters, so that the n points shall be conconical with the points H_ι; and in fact each point H_ι must lie on C^2, but has on it one degree of freedom or its determination on it involves one parametric constant. Here the three conditions imposed upon the principal system are three independent conditions, one on each point; and so the three parameters remaining are independent, one for each point. We cannot, for instance, take H_1 arbitrarily (using two parameters) and afterwards determine H_2H_3 (with still one parameter).†

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* *Conconical;* just as any number of points lying on $\left\{\begin{array}{l}\text{a line}\ldots\ldots\ldots\ldots\\ \text{the circumference of a circle}\end{array}\right\}$ are said to be $\left\{\begin{array}{l}\textit{collinear}\\ \textit{concyclic}\end{array}\right\}$.

† Compare §12, where the principal system has four remaining parameters, which may be used in determing H_1H_2 at will, where H_3 is determined as one of a finite number (two) of points.

Exceptional cases arise when no five of the n points fully determine a conic:

(*a*) If $n-1$ of the n points say P_ι $(\iota=1\ldots n-1)$ are collinear, say on the line p, the conic is p together with any line of the pencil through the remaining point P_n.

(*b*) If the n points are collinear on p, the conic is p together with any line of the net of lines in the plane Π.

These afford illustrations of the remark near the close of §1. That is, in

(*a*) $n-3$ internal conditions are satisfied by the n points, two more than the required $n-5$, which may be interpreted as the conditions that the conic C^2 should decompose into two lines and then that it should be any conic of a certain pencil, that is, should have one parameter; of these two additional internal conditions only the latter relieves the principal system H so that it has $(3+1=)\,4$ parameters; say take H_1 arbitrarily, and $H_2 H_3$ each arbitrarily on the line-pair $(P_n H_1,\, p)$: while in

(*b*) $n-2$ internal conditions are satisfied by the n points, three additional, of which only the two that the (degenerate-) conic should be any conic of a net, *i. e.* should have two parameters, relieve the principal system H, so that it has $(3+2=)\,5$ parameters; say take H_1, H_2 arbitrarily, then H_3 arbitrarily on the line-pair $(H_1 H_2,\, p)$.

Second, $n \leqq 5$. Here the n points are subject to no internal conditions, but may be taken at random. There are then the $n-2$ conditions on the principal system H that the three points H_ι shall be conconical with the n points P_κ; the principal system subject to these conditions has $(6-(n-2)=)\,8-n$ parameters remaining.

Case II. §§4–11.

4. Let the figure E in Ω be two triangles in perspective from a certain centre, *i. e.* for a certain correspondence between the vertices of the triangles, the three lines joining corresponding vertices are concurrent in a point, the centre of perspective. The points of the figure satisfy one condition.

In Π_H the corresponding figure F consists of two triangles $A_1 A_2 A_3$, $B_1 B_2 B_3$, such that the conics C_ι^2 for $\iota=1,\,2,\,3$ pass through a fourth common point I, where C_ι^2 is the conic through the five points $H_1 H_2 H_3 A_\iota B_\iota$. We say, in the plane Π_H the two triangles are in perspective from the centre I. Stated otherwise, there are three conics C_ι^2 of the pencil circumscribing the 4-gon $H_1 H_2 H_3 I$ such that C_ι^2 passes through the points $A_\iota B_\iota$ $(\iota=1,2,3)$.

Taking the two triangles, however, at random, the principal system H is subject to one condition and has five parameters remaining; that is, taking H_1H_2 at random, H_3 must lie on a certain locus, and for a definite position of H_3 on this locus the centre I will be determined uniquely; the second way of looking at the figure shows that I lies on the same locus and has H_3 for its corresponding centre. The two points H_3, I play entirely the same rôle and so may better be written G_1, G_2.

Then given H_1, H_2, A_ι, B_ι $(\iota=1,2,3)$ at random, we are required to determine G_1, G_2 so that certain three conics $C_\iota^2(H_1H_2A_\iota B_\iota)$ shall pass through G_1, G_2 forming a pencil with the base-points $H_1H_2G_1G_2$.

5. Two quadrangles have circumscribing them two pencils of conics, any two of which meet in a group of points. Thus is determined an involution of the points of the plane, *i. e.* every point belongs to one and only one such group. The points of a group are said to be conjugate to one another, or, to specify the involution, conjugate with respect to the two quadrangles. Every group consists of 4, 3, 2, 1 points (the last case is nugatory) according as the two quadrangles have 0, 1, 2, 3 vertices in common.

In this way the two quadrangles $(H_1H_2A_{\iota+1}B_{\iota+1})\,(H_1H_2A_{\iota+2}B_{\iota+2})$[*] determine a quadratic involution of the points of the plane Π; to any point X corresponds a point $Y_{\iota+1,\,\iota+2}$, the fourth point of intersection of the two conics $C^2_{\lambda,\,X}\equiv C^2(H_1H_2A_\lambda B_\lambda X)$, $(\lambda=\iota+1,\iota+2)$.[*] The points X, $Y_{\iota+1,\,\iota+2}$ are conjugate with respect to the two quadrangles or say with respect to $A_{\iota+1}B_{\iota+1}$, $A_{\iota+2}B_{\iota+2}$.

Clearly the points G_1G_2 are conjugate with respect to $A_{\iota+1}B_{\iota+1}$, $A_{\iota+2}B_{\iota+2}$ for $\iota=1,2,3$, because $C^2_{\lambda,\,G_1}$ passes through G_2 $(\lambda=1,2,3)$; conversely, two points in these three ways conjugate are points G_1G_2 (except points XY such that $C^2_{1,\,X}$, $C^2_{2,\,X}$, $C^3_{3,\,X}$ having H_1H_2XY in common yet do not form a pencil; that is, have ∞ points in common and so break up into a common line and three non-concurrent lines; this case arises for any two points XY on H_1H_2 unless A_1B_1, A_2B_2, A_3B_3 are concurrent).

If two points XY are in two of these three ways conjugate, since then $C^2_{\lambda,\,X}$ for $\lambda=1,2,3$ must pass through Y, they are also in the third way conjugate.

6. We find the locus of the points G_1G_2, that is, of the points XY at the

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[*] Throughout the paper the subscripts depending on ι are to be taken *modulo* 3.

same time conjugate with respect to A_1B_1, A_2B_2 and to A_1B_1, A_3B_3, or say of the points X for which Y_{12} and Y_{13} coincide, $Y_{12} \equiv Y_{13} \equiv Y$.

As X describes the range of points lying on a line l passing through H_2, the conic $C^2_{\iota, X} \equiv C^2(H_1H_2A_\iota B_\iota X)$ describes a pencil of conics circumscribing the quadrangle $H_1H_2A_\iota B_\iota$, which (since l passes through H_2, a base-point of the pencil) is projective with the range l; and hence for $\iota = 1, 2, 3$ the three pencils of conics are projective with one another.

Any two corresponding conics $C^2_{1, X}$, $C^2_{2, X}$ intersect in (H_1, H_2 and) two points X, Y_{12} conjugate with respect to A_1B_1, A_2B_2. The locus of the (variable) points of intersection of corresponding conics of the two projective pencils $C^2_{1, X}(H_1H_2A_1B_1)$, $C^2_{2, X}(H_1H_2A_2B_2)$, is a curve of the fourth order having double points at H_1H_2 and passing through A_1, B_1, A_2, B_2; which here consists of the line l, the locus of the point of intersection X and a certain cubic $C^3_{12, l}(H_1^2H_2A_1B_1A_2B_2)$ having a double point at H_1 and passing through the other five points as indicated by the parenthesis, the locus of the other point of intersection Y_{12}. In like manner there is a cubic $C^3_{13, l}(H_1^2H_2A_1B_1A_3B_3)$ the locus of points Y_{13} conjugate to the points X of l.

Let the ray l describe the pencil about H_2; its corresponding $C^3_{12, l}$ will then describe a pencil of cubics whose nine base points are evident, H_1 counting for four; a line through H_2 has one corresponding cubic of this pencil; conversely, a cubic has one corresponding line, since any point on it U quâ Y_{12} by its conjugate point X determines a line $H_2X \equiv l$, and the cubic C^3_{12, H_2X} corresponding to this line H_2X must be (the) one of the pencil passing through U, that is, the original cubic on which U lay. Thus the pencil of rays l through H_2 is projective with the pencil of cubics $C^3_{12}(H_1^2H_2A_1B_1A_2B_2)$, and likewise with the pencil $C^3_{13}(H_1^2H_2A_1B_1A_3B_3)$.

If for a point X, $Y_{12} \equiv Y_{13} \equiv Y$, setting $H_2X \equiv l$, Y quâ Y_{12} lies on $C^3_{12, l}$, Y quâ Y_{13} lies on $C^3_{13, l}$, and so Y is a point of intersection of two corresponding cubics of the pencils C^3_{12}, C^3_{13}. Hence the locus of points XY conjugate at once with respect to A_1B_1, A_2B_2 and to A_1B_1, A_3B_3 is contained in the locus of points of intersection of corresponding cubics of the two projective pencils C^3_{12}, C^3_{13}.

7. This latter locus is a curve of the sixth order, $C^6_1(H_1^4H_2^2A_1^2B_1^2A_2B_2A_2B_3)$. It contains the two lines H_1A_1, H_1B_1. For let X describe the line H_2A_1. Then for X not H_2 or A_1, $C^2_{1, X} \equiv H_2A_1 . H_1B_1$ and $C^2_{2, X} \equiv C^2(H_1H_2A_2B_2X)$, which intersect at X on H_2A_1 and at Y_{12} on H_1B_1. Thus as X describes H_2A_1, Y_{12}

describes H_1B_1 projectively. In fact the cubic of pencil $C^3_{12}(H_1^2H_2A_1B_1A_2B_2)$ which contains one other point of H_1B_1 breaks up into this line H_1B_1 and the conic $C^2_{2, A_1} \equiv C^2(H_1H_2A_2B_2A_1)$; (this conic is locus of points Y_{12} conjugate to the ∞^1 points X adjacent to and surrounding the base-point A_1).

Thus for $l_{(1)} \equiv H_2A_1$,

$$C^3_{12, (1)} \equiv \overline{H_1B_1} . C^2_{2, A_1},$$

and likewise

$$C^3_{13, (1)} \equiv \overline{H_1B_1} . C^2_{3, A_1};$$

and also for $l_{(2)} \equiv H_2B_1$,

$$C^3_{12, (2)} \equiv \overline{H_1A_1} . C^2_{2, B_1},$$

$$C^3_{13, (2)} \equiv \overline{H_1A_1} . C^2_{3, B_1}.$$

The equation of this locus is then by suitable choice of constant factors, writing $C^3 = 0$ for equation of a curve C^3,

$$\frac{C^3_{12, (1)}}{C^3_{12, (2)}} = \frac{C^3_{13, (1)}}{C^3_{13, (2)}};$$

that is;

$$\frac{\overline{H_1B_1} . C^2_{2, A_1}}{\overline{H_1A_1} . C^2_{2, B_1}} = \frac{\overline{H_1B_1} . C^2_{3, A_1}}{\overline{H_1A_1} . C^2_{3, B_1}},$$

or

$$\overline{H_1A_1} . \overline{H_1B_1} (C^2_{2, A_1} C^2_{3, B_1} - C^2_{2, B_1} C^2_{3, A_1}) = 0;$$

say the locus is

$$C_1^6(H_1^4H_2^2A_1^2B_1^2A_2B_2A_3B_3) \equiv \overline{H_1A_1} . \overline{H_1B_1} . C_1^4(H_1^2H_2^2A_1B_1A_2B_2A_3B_3)$$

where

$$C_1^4 \equiv C^2_{2, A_1} C^2_{3, B_1} - C^2_{2, B_1} C^2_{3, A_1}.$$

The locus of points XY conjugate in the three ways is contained in each of the three loci of the points of intersection of corresponding cubics for the three pairs of projective pencils (C^3_{12}, C^3_{13}), (C^3_{12}, C^3_{23}), (C^3_{13}, C^3_{23}). The first as just proved contains the two lines $\overline{H_1A_1} . \overline{H_1B_1}$ and the other two in like manner contain $(\overline{H_1A_2} . \overline{H_1B_2})$, $(\overline{H_1A_3} . \overline{H_1B_3})$ respectively. These lines being in general *distinct*, do not belong to the locus of points XY conjugate in the three ways, which latter is hence a quartic $C^4(H_1^2H_2^2A_1B_1A_2B_2A_3B_3)$, whose equation may be written in three ways,

$$C^4 \equiv C^2_{\iota+1, A_\iota} C^2_{\iota+2, B_\iota} - C^2_{\iota+1, B_\iota} C^2_{\iota+2, A_\iota} = 0 \quad (\iota = 1, 2, 3),$$

putting in evidence that it passes also through the six points D_ι, E_ι, defined as the fourth points of intersection of the two conics

$$(C^2_{\iota+1, A_\iota}, C^2_{\iota+2, A_\iota}), (C^2_{\iota+1, B_\iota}, C^3_{\iota+2, B_\iota}) \text{ respectively.}$$

Here it has, however, been assumed that C_1^6 contains no extraneous factors besides the lines H_1A_1, H_1B_1. This appears as follows: Let Y' be a point of intersection of two corresponding cubics $C^3_{12,\,l}$, $C^3_{13,\,l}$. To Y' quâ $\begin{cases} Y_{12} \text{ on } C^3_{12,\,l} \\ Y_{13} \text{ on } C^3_{13,\,l} \end{cases}$ corresponds a point $\begin{cases} X_{12} \\ X_{13} \end{cases}$ common to $\begin{cases} l,\ C^2_{1,\,Y'},\ C^2_{2,\,Y'} \\ l,\ C^2_{1,\,Y'},\ C^2_{3,\,Y'} \end{cases}$. Now $C^2_{1,\,Y'}(H_1H_2A_1B_1Y')$ has already H_2 in common with l, and hence either (1) $C^2_{1,\,Y'}$ intersects l in only one other point $X_{12} \equiv X_{13}$, in which case the point of intersection Y' of two corresponding cubics is really one of a pair of points Y', $X_{12} \equiv X_{13}$ conjugate at once with respect to A_1B_1, A_2B_2 and to A_1B_1, A_3B_3; or (2) $C^2_{1,\,Y'}$ contains the ray l completely (*i. e.* $C^2_{1,\,Y'}$ degenerates into a line-pair), in which case we do do not at all know that $X_{12}{}^0$ common to l and $C^2_{2,\,Y'}$ coincides with X_{13} common to l and $C^2_{3,\,Y'}$, and in fact in general it does not.

Hence if the two corresponding cubics do intersect in a point Y' not one of a pair of points in the two ways conjugate, they must correspond to a line l which a conic of pencil C_1^2 contains entirely, *i. e.* $l \equiv H_2A_1$, H_2B_1 or H_2H_1. The cubics corresponding to $H_2A_1\,(H_2B_1)$ have been shown to have in common the extraneous factor $H_1B_1\,(H_1A_1)$. The cubics corresponding to H_1H_2 must pass through the intersection of A_1B_1, A_2B_2 and of A_1B_1, A_3B_3 respectively; they do not coincide and so have in common no extraneous factor. The assumption referred to is thus justified.

8. Thus the locus of the points G_1G_2 such that the conics $C^2_{\iota,\,G_1}$ $(\iota = 1,\ 2,\ 3)$ form a pencil with the base-points $H_1H_2G_1G_2$ is a binodal quartic

$$\mathbf{C}^4 \equiv C^4\,(H_1^2H_2^2A_\iota B_\iota D_\iota E_\iota) \qquad (\iota=1,2,3),$$

on which the point-pairs G_1G_2 form a quadratic involution.

In case $H_1H_2A_1B_1A_2B_2$ are conconical, *i. e.* $C^2_{2,\,A_1} \equiv C^2_{2,\,B_1}$, then the locus quartic $\mathbf{C}^4$ contains this conic say $\mathbf{C}^2_{12}$, which is divided in an involution of points G_1G_2 by the pencil of conics C_3^2.

If further $H_1H_2A_1B_1A_3B_3$ are conconical on $\mathbf{C}^2_{13}$, this is likewise a factor of the locus quartic $\mathbf{C}^4$, and $\mathbf{C}^4 \equiv \mathbf{C}^2_{12}\mathbf{C}^2_{13}$.

If still further $H_1H_2A_2B_2A_3B_3$ are conconical on $\mathbf{C}^2_{12}$, this is in like manner a factor of $\mathbf{C}^4$; this can be if $\mathbf{C}^2_{23} \equiv \mathbf{C}^2_{31} \equiv \mathbf{C}^2_{12} \equiv$ say $\mathbf{C}^2_{123}$, *i. e.* if the eight points $H_1H_2A_\iota B_\iota\,(\iota=1,2,3)$ are conconical, or if in any way by the breaking up of the conics $\mathbf{C}^2_{12}\mathbf{C}^2_{13}$ contains $\mathbf{C}^2_{23}$ as a factor; unless this is so, $\mathbf{C}^4$ must be indeterminate.

These special cases may well be considered independently and more fully.

9. In the plane Π let three (non-coincident) conics

$$\mathbf{C}^{2'}_{\iota+1,\,\iota+2}(H_1H_2A'_{\iota+1}B'_{\iota+1}A'_{\iota+2}B'_{\iota+2}) \qquad (\iota=1,2,3)$$

have a common chord H_1H_2, then their three other chords $A'_\iota B'_\iota$ $(\iota=1,2,3)$ are concurrent* say at G'_2.

Transform the plane into itself by a quadratic transformation with $H_1H_2G_1$ (where G_1 is chosen at random in the plane) as principal points. The resulting theorem may be expressed thus: Let three (non-coincident) conics

$$\mathbf{C}^{2}_{\iota+1,\,\iota+2}(H_1H_2A_{\iota+1}B_{\iota+1}A_{\iota+2}B_{\iota+2}) \qquad (\iota=1,2,3)$$

exist. Then taking any point G_1 of the plane, the three conics $C^2_{\iota,\,G_1}$ are concurrent at G_2 forming a pencil.

Thus is established a quadratic involution of the points G_1G_2 of the plane, which must be identical with the quadratic involution of points conjugate with regard to A_1B_1, A_2B_2 conconical with H_1H_2 on $\mathbf{C}^2_{12}$. ($\mathbf{C}^4$ is indeterminate.)

If H_1H_2 are the two circular points at infinity, the quadratic involution is that of inverse points with reference to the radical centre and the orthogonal circle of the three circles. The known generalization of this *inversion* or *transformation by reciprocal radii vectores* is then to be considered, since by it these special cases become clear.

10. Given a point C and a conic C^2_C. Two points $R'R''$ are said to be *inverse* with respect to the centre C and the conic C^2_C (or briefly, with respect to C and C^2_C) when R' and R'' are collinear with C and are conjugate with respect to C^2_C, *i. e.* the segment $R'R''$ is divided harmonically by C^2_C.

Let H_1H_2 be the chord of contact of tangents through C to C^2_C. The points $W_1 \equiv H_1R'.H_2R''$, $W_2 \equiv H_1R''.H_2R'$ are called the *anti-points* of R' and R'' with respect to H_1H_2; R' and R'' being inverse with respect to C and C^2_C, the anti-points lie on C^2_C.† Hence the inverse of a line H_1W where W is on C^2_C is the line H_2W. Any line through C of course inverts into itself. Points on H_1H_2, CH_1, CH_2 invert into C, H_1, H_2 respectively, or really into elements of direction at those points. This inversion is then a quadratic transformation of the plane with CH_1H_2, CH_2H_1 as corresponding principal points. A conic through H_1H_2 inverts into a conic through H_1H_2. In particular, a conic through two inverse points $R'R''$ intersecting C_C^2 in T and U, $C^2(H_1H_2R'R''TU)$ inverts into a $C^2(H_1H_2R''R'TU)$, *i. e.* into itself; any two points $S'S''$ on it collinear with C are inverse; in particular, CT is tangent to the conic at T, which may be expressed by the theorem that C_C^2 is the locus of the points of tangency of tangents through C to all conics (which invert into themselves) of a pencil through a pair of inverse points and H_1H_2. Any two pairs of inverse points are conconical with H_1H_2.

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* Salmon's Conic Sections, 6th ed., §266.

† Let the line $CR'R''$ intersect H_1H_2 in U and C^2_C in V_1V_2. Then on this line CUV_1V_2 and $R'R''V_1V_2$ are two harmonic ranges. The harmonic pencil $H_1(CUV_1V_2)$ gives on the conic the harmonic range $H_1H_2V_1V_2$, H_1C being tangent at H_1. Let H_1R' intersect C^2_C in (H_1 and also) W. The pencil $W(H_1H_2V_1V_2)$ is harmonic and cuts $CR'R''$ in the harmonic range $R'(WH_2.R'R'')V_1V_2$, where since $R'R''V_1V_2$ is harmonic, $WH_2.R'R'' \equiv R''$, *i. e.* WH_2R'' are collinear; $\therefore$ in fact H_1R' and H_1R'' intersect at (W or) W_1 *on the conic*, and likewise for W_2.

Conversely, given any six conconical points $H_1H_2A_1B_1A_2B_2$ on $\mathbf{C}_{12}^2$, take $C \equiv A_1B_1 . A_2B_2$ and C_C^2 that conic of pencil tangent to CH_1, CH_2 at H_1, H_2 which divides A_1B_1 harmonically; then A_1B_1 are inverse with respect to C and C_C^2 and, by what precedes, also A_2B_2 are inverse.

Two conics C_1^2, C_2^2 each through H_1H_2 and a pair of inverse points A_1B_1, A_2B_2 intersect in two points X_1Y_{12}, and since each conic inverts into itself, these points are a pair of inverse points.* That is, any two points XY_{12} conjugate with respect to A_1B_1, A_2B_2 in the sense of §5, *where* $H_1H_2A_1B_1A_2B_2$ *are conconical on* $\mathbf{C}_{12}^2$, are inverse points in the inversion determined by H_1H_2 as base-points and by A_1B_1, A_2B_2 as two pairs of inverse points; and conversely. Notice, however, as exception, that any point on $\mathbf{C}_{12}^2$ is with respect to A_1B_1, A_2B_2 conjugate to every other one.

11. Let $H_1H_2A_1B_1A_2B_2$ be conconical on $\mathbf{C}_{12}^2$, thus determining a certain centre C and conic C_C^2 of an inversion. Then the binodal quartic $\mathbf{C}^4$ degenerates into $\mathbf{C}_{12}^2$ and a conic of pencil $C_3^2(H_1H_2A_3B_3)$. The points conjugate with respect to A_1B_1, A_2B_2 are by §10 (except the point-pairs lying on $\mathbf{C}_{12}^2$) inverse. This conic of pencil C_3^2 quâ locus of point-pairs G_1G_2 conjugate in three ways must invert into itself; that is, it is that conic of pencil which joins the two pairs of inverse points A_3A_3', B_3B_3', say $C^2(H_1H_2A_3B_3A_3'B_3')$.†

This conic is determinate unless A_3B_3 are themselves inverse points. In fact in this case $\mathbf{C}^4$ is indeterminate, for $A_\iota B_\iota\,(\iota=1,2,3)$ being inverse points, any pair of inverse points XY is conconical with $H_1H_2A_\iota B_\iota\,(\iota=1,2,3)$, and so is a pair of points G_1G_2. There are two cases of this kind; (*a*) the six points $A_\iota B_\iota$ are conconical with H_1H_2 on one conic $\mathbf{C}^2_{123}$, and the chords $A_\iota B_\iota$ are concurrent at C; (*b*) the six points are the points of intersection by pairs of three conics $\mathbf{C}^2_{12}$, $\mathbf{C}^2_{13}$, $\mathbf{C}^2_{23}$ having the common chord H_1H_2.

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* Whence easily a proof of the theorem referred to in §9.

† Observe that $\left\{\begin{matrix} A_3' \\ B_3' \end{matrix}\right.$ may be defined as fourth intersection of $\left\{\begin{matrix} C_{1,A_3}^2, & C_{2,A_3}^2; \\ C_{1,B_3}^2, & C_{2,B_3}^2 \end{matrix}\right.$ that is, $A_3', B_3' \equiv D_3, E_3$, in agreement with theory of §7. Here since $C_{1,A_2}^2 \equiv C_{1,B_2}^2 \equiv C_{2,A}^2 \equiv C_{2,B}^2 \equiv \mathbf{C}_{12}^2$, $D_1E_1D_2E_2$ are indeterminate.

If A_3B_3 are not inverse but yet conconical with $H_1H_2A_1B_1$ on $\mathbf{C}^2_{13}$, then $\mathbf{C}^2_{13}$ quâ C^2_1 inverts into itself and so coincides with $C^2(H_1H_2A_3B_3A_3'B_3')$, and the quartic $\mathbf{C}^4$ breaks up into the two conics $\mathbf{C}^2_{12}$, $\mathbf{C}^2_{13}$. On $\left\{\begin{matrix}\mathbf{C}^2_{12}\\ \mathbf{C}^2_{13}\end{matrix}\right.$ the point-pairs G_1G_2 are given as points of intersection with conics of the pencil $\left\{\begin{matrix}C^2_3\\ C^2_2\end{matrix}\right.$. Observe that $H_1H_2A_2B_2A_3B_3$ cannot be conconical, or A_3B_3 would be inverse, unless the *eight* points are conconical on $\mathbf{C}^2_{123}$ and the chords $A_\iota B_\iota$ are not concurrent; in this latter case $\mathbf{C}^4 \equiv \mathbf{C}^2_{123}\mathbf{C}^2_{123}$ and any point of $\mathbf{C}^2_{123}$ is in three ways conjugate with every other point.

Application of the results of Case II. §§12–18.

12. Given six points $K_1 \ldots . K_6$ grouped in two ways (1)(2) into triples of pairs, $A_{\iota(1)}B_{\iota(1)}$, $A_{\iota(2)}B_{\iota(2)}\,(\iota=1,2,3)$. Taking H_1H_2 at random, two definite quartics $\mathbf{C}^4_{(1)}$, $\mathbf{C}^4_{(2)}$ of the net[*] of quartics $C^4(H_1^2H_2^2K_1 \ldots . K_6)$ are the loci of point pairs $G_{1(1)}G_{2(1)}$, $G_{1(2)}G_{2(2)}$ for the two correspondences or groupings (1), (2). These two quartics intersect in the base-points and also in $(4^2-(2.2^2+6)=)$ two other points I_1I_2. Let the point-pairs G_1G_2 for the correspondence

$$\left\{\begin{matrix}(1)\\ (2)\end{matrix}\right. \text{ be on} \left\{\begin{matrix}\mathbf{C}^4_{(1)} & I_1J_{1(1)} \text{ and } I_2J_{2(1)}\\ \mathbf{C}^4_{(2)} & I_1J_{1(2)} \text{ and } I_2J_{2(2)}\end{matrix}\right. .$$

Then the six points $K_1 \ldots . K_6$ have the perspective relations (1) and (2) in the plane $\Pi_{H_1H_2I_1}$ from the centres $J_{1(1)}$, $J_{1(2)}$; that is, the conics

$$\left\{\begin{matrix}C^2_{\iota(1)}(H_1H_2I_1A_{\iota(1)}B_{\iota(1)})\\ C^2_{\iota(2)}(H_1H_2I_1A_{\iota(2)}B_{\iota(2)})\end{matrix}\right. \text{ for } \iota=1,\ 2,\ 3 \text{ from a pencil through} \left\{\begin{matrix}J_{1(1)} \text{ on } \mathbf{C}^4_{(1)}\\ J_{1(2)} \text{ on } \mathbf{C}^4_{(2)}\end{matrix}\right. .$$

They have the same perspective relations in the plane $\Pi_{H_1H_2I_2}$ from the centres $J_{2(1)}J_{2(2)}$.

Thus, taking H_1H_2 at random, two and only two points I_1I_2 can be found such that in each of the planes $\Pi_{H_1H_2I_1}$, $\Pi_{H_1H_2I_2}$ the six points $K_1 \ldots . K_6$ shall have the two perspective relations according to the given correspondences (1), (2).

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[*] Net; for a quartic is fully determined by 14 conditions, and here there are $2.3+6=12$.

13. In particular, separating the six points into two triangles $K_1K_2K_3$, $K_4K_5K_6$, there are three say *cyclic* correspondences,

$$\begin{pmatrix} K_1K_2K_3 \\ K_4K_5K_6 \end{pmatrix}, \begin{pmatrix} K_1K_2K_3 \\ K_5K_6K_4 \end{pmatrix}, \begin{pmatrix} K_1K_2K_3 \\ K_6K_4K_5 \end{pmatrix},$$

say correspondences (1)(2)(3).

In a plane the theorem holds; if two triangles are perspective according to two correspondences of a cyclic set of three, then they are also perspective according to the third correspondence of that set.* By a quadratic transformation this is shown to hold also in a plane $\Pi_{H_1H_2H_3}$.

Now these two triangles are in perspective in the two ways (1)(2) in each of the planes $\Pi_{H_1H_2I_1}$, $\Pi_{H_1H_2I_2}$; hence in each plane they must be in perspective also in the third way (3), and hence the corresponding locus quartic $\mathbf{C}^4_{(3)}$ must pass through I_1, I_2.

The three locus quartics $\mathbf{C}^4_{(1)}\mathbf{C}^4_{(2)}\mathbf{C}^4_{(3)}$ of point-pairs G_1G_2 for three cyclic correspondences (1)(2)(3) of the vertices of two arbitrary triangles $K_1K_2K_3$, $K_4K_5K_6$ form a pencil in the net of quartics $C^4(H_1^2H_2^2K_1 \ldots . K_6)$.

14. The six points $K_1 \ldots . K_6$ may be divided in 10 ways into a pair of triangles; the vertices of two triangles may be arranged in triples of pairs of corresponding vertices in 6 ways (arising from the six permutations of the three vertices of one of the triangles); any grouping of the six points into triples of pairs, $A_\iota B_\iota$, (§12) gives a correspondence between the vertices of 4 pairs of triangles $\begin{pmatrix} A_1A_2A_3 \\ B_1B_2B_3 \end{pmatrix}, \begin{pmatrix} A_1A_2B_3 \\ B_1B_2A_3 \end{pmatrix}, \begin{pmatrix} A_1B_2A_3 \\ B_1A_2B_3 \end{pmatrix}, \begin{pmatrix} A_1B_2B_3 \\ B_1A_2A_3 \end{pmatrix}$. Thus the six points may be grouped in $\frac{10.6}{4} = 15$ ways into triples of pairs of points. Every such grouping has a corresponding locus quartic $\mathbf{C}^4$ belonging to the net $C^4(H_1^2H_2^2K_1 \ldots . K_6)$.

From §13 and the remark above it is clear that every such locus quartic belongs in four ways to a pencil of three such quartics. I hope at another time to discuss more fully this system of 15 quartics in the net $C^4(H_1^2H_2^2K_1 \ldots . K_6)$.

15. Suppose that for a grouping $A_\iota B_\iota$ of the six given points $K_1 \ldots . K_6$ the lines $A_\iota B_\iota$ ($\iota = 1, 2, 3$) are concurrent say at L; that is, that the corresponding

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* Von Staudt, Geometrie der Lage, p. 125; Rosanes, p. 550, and Schröter, p. 555 of the Math. Annalen, II.

perspective relation exists *in the simple plane* Π. Then clearly taking H_1H_2 at random and any point X on H_1H_2, the perspective relation exists identically in the plane $\Pi_{H_1H_2X}$ or $\Pi_{C^2(H_1H_2X)}$ from the centre L, for a *line* $C^2(H_1H_2X)$ in the plane $\Pi_{C^2(H_1HX)}$ consists of H_1H_2X itself and a line of the simple plane Π, and in particular the conics $C^2_{\iota, X}$ from a pencil consisting of the common line H_1H_2X and the pencil of lines $A_\iota B_\iota$ through L. Thus, if for a grouping of the points $K_1 \dots . K_6$ into a triple of pairs, the perspective relation holds in the simple plane Π, then the locus quartic consists of the line H_1H_2 itself and the cubic $C^3(H_1H_2K_1 \dots . K_6L)$, *i. e.* that cubic of the pencil through the eight points $H_1H_2K_1 \dots . K_6$ which passes also through the centre of perspective; this cubic is the real locus of point-pairs G_1G_2 for this grouping (the point L alone corresponding to all points on H_1H_2, and conversely), and one may speak in this case of the locus cubic.

If several perspective relations of the points K exist in the simple plane Π (that is, if for several groupings the lines $A_\iota B_\iota$ are concurrent), since the locus cubics for these groupings all pass through a point say H_3, the ninth base-point of the pencil of cubics through $H_1H_2K_1 \dots . K_6$, the same perspective relations exist in the plane $\Pi_{H_1H_2H_3}$. Or more compactly, *whatever perspective relations among the six points K exist in the simple plane* Π *exist likewise in the plane* $\Pi_{H_1H_2H_3}$ *or* $\Pi_{C^2(H_1H_2H_3)}$, *where* $H_1H_2H_3$ *with* $K_1 \dots . K_6$ *make up the nine base-points of a pencil of cubics.*

The converse is true; for if a perspective relation exist in $\Pi_{H_1H_2H_3}$, the corresponding locus quartic must pass through H_3; now the pencil of quartics of the net $C^4(H_1^2H_2^2K_1 \dots . K_6)$ passing through H_3 consists of the common line H_1H_2 and the pencil of cubics through the nine base-points; thus this particular locus quartic must contain the line H_1H_2, and the perspective relation must hold in the plane $\Pi_{H_1H_2X}$ (X being any point on H_1H_2), or, what is identically the same, in the simple plane Π.

16. I give two illustrative cases in which the theorems of §§15, 18 have application: (*a*) Two triangles may be in 1, 2, 3, 4 or 6 ways in perspective; this last case does not occur when both triangles are real, the simplest illustration being an equilateral triangle and the triangle with vertices at the two circular points at infinity and at the centre of the circle circumscribing the first triangle.*

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* Multiply perspective triangles were first discussed by Rosanes, Schröter, Math. Annalen II; later by Kantor, Ueber die Configurationen (3, 3) mit den Indices 8, 9, etc.; Sitzungsber. d. Wien. Akad. II Abtheilung, 1881, LXXXIV, 915–932; and by Hess, Beiträge zur Theorie der mehrfach perspectiven Dreiecken und Tetraeder, Math. Ann. 1886, XXVIII, 167–260.

(*b*) Six points may form a *Clebsch's six-gon* with the property that two triangles formed in any way with these six points as vertices are four-ply perspective.[*] The simplest illustration is the six-gon whose six vertices are the five vertices of a regular pentagon and its centre as sixth vertex.

17. If we transform the plane Π into itself by a quadratic transformation with any three points $G_{(1)}$, $G_{(2)}$, $G_{(3)}$ as self-corresponding principal points, the net of conics $C^2(H_1 H_2 H_3)$ transforms into the net of trinodal quartics $C^4(G^2_{(1)1} G^2_{(1)2} G^2_{(1)3} G_{(0)1} G_{(0)2} G_{(0)3})$ (where H_ι transforms into $G_{(0)\iota}$); we may say that the plane $\Pi_{C^2(H_\iota)}$ transforms into the plane $\Pi_{C^4(G^2_{(1)\iota} G_{(0)\iota})}$.

The theorem of §15 was, whatever perspective relations hold for the six points $K_1 \dots . K_6$ in the simple plane Π hold also in the plane $\Pi_{C^2(H_\iota)}$, where the nine points $H_\iota K_\kappa$[†] are the nine base-points of a pencil of cubics C^3; and conversely.

By a quadratic transformation with the arbitrary principal points $G_{(1)\iota}$, this theorem becomes: Whatever perspective relations hold for the six points $K'_1 \dots . K'_6$ in the plane $\Pi_{C^2(G_{(1)\iota})}$ hold likewise in the plane $\Pi_{C^4(G^2_{(1)\iota} G_{(0)\iota})}$ where the nine points $G_{(0)\iota} K'_\kappa$ are the 9 base-points of a pencil of $C^6(G^3_{(1)\iota})$; and conversely.

By mathematical induction, using successive quadratic transformations, the general theorem is proved:

Given six points $K_1 \dots . K_6$ and $3(n+1)$ points $G_{(\lambda)\iota} \left({\lambda = 0, 1, \dots n \atop \iota = 1, 2, 3} \right)$ of which all are arbitrary except $G_{(0)3}$, which is determined (in general uniquely) by the condition that the nine points $G_{(0)\iota} K_\kappa$ shall be the 9 base-points of a pencil of $C^{3.2^n} \underset{\lambda = n \dots . 1}{(G^{3.2^{\lambda-1}}_{(\lambda)\iota})}$;[‡] then whatever perspective relations among the six points K hold in the plane

$$\Pi_{C^{2^n} \underset{\lambda = n \dots . 1}{(G^{2^\lambda - 1}_{(\lambda)\iota})}}$$

hold also in the plane

$$\Pi_{C^{2^{n+1}} \underset{\lambda = n \dots . 1}{(G^{2\lambda}_{(\lambda)\iota}\, G_{(0)\iota})}} \equiv \Pi_{C^{2^{n+1}} \underset{\lambda = n \dots . 0}{(G^{2\lambda}_{(\lambda)\iota})}},$$

and conversely.

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[*] This six-gon discovered by Clebsch, Math. Ann. 1871, IV, 284, was further discussed, from the standpoint of ikosaedron investigations, by Klein and by Hess, under the name "Das zehnfach Brianchon'sche Sechseck," and more fully by Schröter, "Das Clebsch'sche Sechseck," Math. Ann. 1887, XXVIII, 457–482, to whom the latter probably permanent name is due.

[†] Throughout this number and the following ones $\iota = 1, 2, 3$, $\kappa = 1, 2 \dots . 6$.

[‡] $3 \sum_{\lambda=1}^{\lambda=n} (3.2^{\lambda-1})^2 + 9 \equiv (3.2^n)^2$.

18. Given $K_1 \dots . K_6$ with certain perspective relations in the simple plane Π. Define any number of successive triples of points $H_{(\lambda)\iota}$ as follows:

Let $K_K H_{(0)\iota}$ be the nine base-points of a pencil of C^3,
and then $K_K H_{(1)\iota}$ " " " " " $C^6(H^3_{(0)\iota})$,
and then $K_K H_{(2)\iota}$ " " " " " $C^{12}(H^6_{(0)\iota} H^3_{(1)\iota})$,
and so on; that is, in general, let

$$K_K H_{(m)\iota} \text{ be the nine base-points of a pencil of } C^{3\cdot 2^m}\big(\underset{\lambda=0.1\dots.m-1}{H^{3\cdot 2^{m-\lambda-1}}_{(\lambda)\iota}}\big).$$

Each triple depends upon all the preceding triples; these being known, two points of each triple determine (in general uniquely) the third point.

The preceding general proposition, successively applied, shows that: Whatever perspective relations among the six points $K_1 \dots . K_6$ hold in the simple plane Π hold likewise in the plane $\Pi_{C^2(H_{(0)\iota})}$, and hence in the plane $\Pi_{C^4(H^2_{(0)\iota} H_{(1)\iota})}$, and hence in the plane $\Pi_{C^8(H^4_{(0)\iota} H^2_{(1)\iota} H_{(2)\iota})}$, and so on and in general in the plane

$$\Pi_{C^{2^{m+1}}}\big(\underset{\lambda=0.1\dots.m}{H^{2^{m-\lambda}}_{(\lambda)\iota}}\big).^*$$

Conversely, if a perspective relation among the six points $K_1 \dots . K_6$ holds in the plane with any one of these nets of curves $\Pi_{C^{2^{m+1}}}\big(\underset{\lambda=0.1\dots.m}{H^{2m-\lambda}_{(\lambda)\iota}}\big)$, then it holds in all, and so also in the simple plane Π.

YALE, NEW HAVEN, CONN., March 20, 1888.

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*In general case, for comparison of the theorems, set $H_{(\lambda)\iota} \equiv G_{(m-\lambda)\iota}$.

Sur l'orientation des systèmes de droites.

Par M. G. Humbert.

I.—*Théorèmes fondamentaux.*

1. Laguerre a fait connaître, dans le Bulletin de la Société philomatique, plusieurs propositions géométriques très simples, relatives aux directions des systèmes de droites dans le plan, et il en a déduit des conséquences nombreuses et importantes: nous avons eu nous même l'occasion, dans un mémoire sur le théorême d'Abel, de retrouver analytiquement ces propositions et de leur donner une certaine extension; notre but est maintenant de démontrer un principe très général, auquel peuvent se rattacher toutes les propriétés énoncées jusqu'ici sur les directions des systèmes de droites, et qui se prête aisément à des applications nouvelles.

A cet effet, nous commencerons par présenter sous une forme nouvelle une notion importante, introduite dans la Géométrie par Laguerre, celle de l'*orientation* d'un système de droites. La définition donnée par Laguerre est la suivante.

Soient, dans un plan, deux systèmes de n droites, A et A'; prenons arbitrairement un axe fixe, H, dans ce plan: si la somme des angles que font avec l'axe fixe les droites du système A est égale, à un multiple de π près, à la somme analogue pour les droites du système A', on dit que les systèmes A et A' ont même *orientation:* cette propriété est évidemment indépendante du choix de l'axe fixe H dans le plan: elle ne dépend que des directions des droites considérées.

2. Cette définition peut être transformée et précisée, au point de vue analytique, comme il suit.

Menons par l'origine des parallèles :

$$y - a_1 x = 0,\ y - a_2 x = 0, \ldots\ldots y - a_n x = 0$$

et

$$y - a'_1 x = 0, \qquad \ldots\ldots y - a'_n x = 0,$$

aux n droites de chacun des systèmes A et A'; soient $\alpha_1 \ldots\ldots \alpha_n$; $\alpha'_1 \ldots\ldots \alpha'_n$, les angles de ces droites avec $0x$, les axes étant supposés rectangulaires. On a

$$\alpha_\kappa = \text{arc tg } a_\kappa, \quad \alpha'_\kappa = \text{arc tg } a'_\kappa,$$

d'où :

$$e^{2i\alpha_\kappa} = \cos(2 \text{ arc tg } a_\kappa) + i \sin(2 \text{ arc tg } a_\kappa),$$

c. à d.

$$e^{2i\alpha_\kappa} = \frac{1 - a_\kappa^2}{1 + a_\kappa^2} + i\,\frac{2a_\kappa}{1 + a_\kappa^2} = \frac{(1 + ia_\kappa)^2}{1 + a_\kappa^2} = \frac{i - a_\kappa}{i + a_\kappa},$$

et par suite

$$e^{2i(\alpha_1 + \alpha_2 + \ldots\ldots + \alpha_n)} = \frac{(i - a_1)(i - a_2) \ldots\ldots (i - a_n)}{(i + a_1)(i + a_2) \ldots\ldots (i + a_n)}.$$

Soit posé maintenant

$$(y - a_1 x)(y - a_2 x) \ldots\ldots (y - a_n x) = f(x, y),$$

$$(y - a'_1 x) \qquad \ldots\ldots (y - a'_n x) = \varphi(x, y),$$

il vient

$$e^{2i(\alpha_1 + \alpha_2 + \ldots\ldots + \alpha_n)} = \frac{f(1, i)}{f(-1, i)}$$

et

$$e^{2i(\alpha'_1 + \ldots\ldots + \alpha'_n)} = \frac{\varphi(1, i)}{\varphi(-1, i)}.$$

Si donc les deux systèmes A et A' ont même orientation, c. à d. si l'on a, d'après la définition

$$\alpha_1 + \ldots\ldots + \alpha_n = \alpha'_1 + \ldots\ldots + \alpha'_n + h\pi,$$

on aura

$$\frac{f(1, i)}{f(-1, i)} = \frac{\varphi(1, i)}{\varphi(-1, i)}.$$

On peut donc considérer comme définissant l'orientation d'un système de droites issues de l'origine, représenté par l'équation homogène $f(x, y) = 0$, le rapport $\frac{f(1, i)}{f(-1, i)}$. Si les droites ne passent pas toutes par l'origine, et si $f(x, y, z) = 0$ est l'équation de leur ensemble, l'orientation sera définie par le rapport $\frac{f(1, i, 0)}{f(-1, i, 0)}$ et, plus généralement encore, si $f(x, y, z) = 0$ est l'équation d'une courbe algébrique quelconque, le rapport $\frac{f(1, i, 0)}{f(-1, i, 0)}$ définira l'orientation du système des directions asymptotiques de cette courbe.

3. Cela posé, considérons dans un plan un système variable de n droites, dont l'équation dépend rationnellement d'un paramètre λ, et soit

$$\lambda^p A + \lambda^{p-1} B + \ldots . + \lambda L + M = 0, \tag{1}$$

cette équation. L'orientation du système est définie par le quotient:

$$\omega = \frac{\lambda^p A(1, i, 0) + \lambda^{p-1} B(1, i, 0) + \ldots . + M(1, i, 0)}{\lambda^p A(-1, i, 0) + \lambda^{p-1} B(-1, i, 0) + \ldots . + M(-1, i, 0)}.$$

Pour que ω soit indépendant de λ, c. à d. pour que le système variable ait une orientation fixe, il faut et il suffit que l'on ait:

$$\frac{A(1, i, 0)}{A(-1, i, 0)} = \frac{B(1, i, 0)}{B(-1, i, 0)} = \ldots . = \frac{M(1, i, 0)}{M(-1, i, 0)}. \tag{2}$$

Ces conditions peuvent s'interpréter géométriquement d'une manière très élégante: les racines de l'équation $\lambda^p A(1, i, 0) + \ldots . + M(1, i, 0) = 0$ sont en effet les valeurs de λ qui correspondent aux systèmes compris dans l'équation (1) et contenant une droite qui passe par le point cyclique $I(x = 1,\ y = i,\ z = 0)$; de même les racines de l'équation $\lambda^p A(-1, i, 0) + \ldots . + M(-1; i, 0) = 0$ sont les valeurs de λ qui correspondent aux systèmes contenant une droite qui passe par le point cyclique $J(x = -1,\ y = i,\ z = 0)$; les conditions (2) expriment que ces équations ont les mêmes racines, c. à d. que tout système contenant une droite (et généralement $\varkappa$ droites) passant par I, contient aussi une droite (et généralement $\varkappa$ droites) passant par J. De la cette conclusion fondamentale:

Théorème.—Pour qu'un système variable de n droites, dont l'équation contient rationnellement un paramètre, conserve dans le plan une orientation fixe, il faut et il suffit que lorsqu'une ou plusieurs des droites du système viennent à passer par un des points cycliques, d'autres droites du système en même nombre, passent au même instant par l'autre point cyclique.

Plus généralement, si l'equation (1) est celle d'une famille de courbes algébriques, on peut énoncer la proposition suivante:

Soit une famille de courbes algébriques, dont l'équation contient rationnellement un paramètre: pour que l'orientation du système des directions asymptotiques de chacune de ces courbes soit constante, il faut et il suffit que toutes les courbes de la famille qui passent par un des points cycliques du plan, passent en même temps par l'autre.

4. On peut faire de ces principes des applications nombreuses. Considérons d'abord le cas où le paramètre λ figure au premier degré dans l'équation d'une famille de courbes ; ces courbes appartiennent alors à un même faisceau ponctuel,

$$f(x, y, z) + \lambda\varphi(x, y, z) = 0.$$

L'orientation du système des directions asymptotiques d'une des courbes précédentes dépend du coefficient

$$\omega = \frac{f(1, i, 0) + \lambda\varphi(1, i, 0)}{f(-1, i, 0) + \lambda\varphi(-1, i, 0)}$$

et de cette expression résulte immédiatement ce théorème :

Si deux courbes algébriques de degré n sont telles que leurs systèmes respectifs d'asymptotes aient même orientation, le système des asymptotes de toute autre courbe de degré n, passant par les points d'intersection des deux premières, aura même orientation que chacun des deux systèmes primitifs.

Si la courbe $\varphi = 0$ passe par les points cycliques du plan, $\varphi(1, i, 0)$ et $\varphi(-1, i, 0)$ sont nuls ; par suite :

Si deux courbes de même degré rencontrent aux mêmes points une courbe algébrique quelconque passant par les points cycliques du plan, les deux systèmes formés par leurs directions asymptotiques ont même orientation.

Comme cas particulier de cette proposition, on retrouve un théorême important, dû à Laguerre, et qu'on obtient en supposant que la courbe algébrique considérée devienne un cercle :

Si l'on groupe deux à deux, d'une manière quelconque, sur n droites, les 2n points communs à un cercle et à une courbe algébrique d'ordre n, l'orientation de chacun des systèmes de n droites ainsi obtenus est la même que celle des asymptotes de la courbe.

II.— *Orientation de certains systèmes de tangentes.*

5. En transformant par polaires réciproques quelques unes des propositions qui précèdent, on arrive à des théorêmes intéressants sur l'orientation du système des tangentes qu'on peut mener d'un point à une courbe ; ainsi, la proposition qui termine le n° 3 donne lieu à la suivante :

Soit une famille de courbes dont l'équation tangentielle contient rationnellement un paramètre : pour que l'orientation du système des tangentes qu'on peut mener d'un point fixe, 0, *à chacune de ces courbes demeure constante, il faut et il suffit que toutes* *les courbes de la famille qui touchent une des droites isotropes issues de* 0 *touchent l'autre droite isotrope issue de ce point.*

En particulier, si les courbes considérées appartiennent à un même faisceau tangentiel, une seule de ces courbes touchera une droite isotrope issue de 0; si elle touche en même temps l'autre droite isotrope, le point 0 sera un foyer de cette courbe. Donc:

Soit un faisceau tangentiel de courbes algébriques de classe n; par un foyer f de l'une d'elles menons les n tangentes à l'une quelconque des autres: tous les systèmes ainsi obtenus à partir du point f ont même orientation. Reciproquement, *si un point f jouit de cette propriété, c'est le foyer de l'une des courbes du faisceau.*

Si deux des courbes du faisceau sont homofocales, toutes les courbes du faisceau ont les mêmes foyers; l'une d'elles se décompose en une courbe de classe $n-2$ et en deux points, qui sont les points cycliques du plan. Un point quelconque du plan peut être considéré comme un foyer de ce système de deux points, il résulte de là, par l'application du théorème précédent, que:

Les deux systèmes formés par les tangentes que l'on peut mener d'un point quelconque à deux courbes homofocales de même classe ont même orientation; ou encore:

Le système des tangentes menées d'un point quelconque à une courbe algébrique de classe n, et le système des droites qui joignent le même point aux n foyers réels de la courbe ont même orientation. [Laguerre.]

En combinant ce résultat avec le précédent, on arrive à une proposition simple relative au lieu des foyers des courbes d'un même faisceau tangentiel.

Le lieu des foyers des courbes d'un faisceau tangentiel déterminé par deux courbes A et B, de classe n, est une courbe telle que si l'on joint un de ses points aux n foyers réels de A et aux n foyers réels de B, les deux systèmes de droites ainsi obtenus aient même orientation.

6. Nous reviendrons plus loin, avec quelques détails sur les conséquences géométriques de ce théorème; auparavant, nous ferons une application des principes précédents à la solution d'un problème qui parait présenter un certain intérêt.

Ce problème est le suivant:

Trouver toutes les courbes algébriques telles que le système des tangentes qu'on *peut mener d'un point à l'une d'elles ait une orientation fixe indépendante de la position de ce point dans le plan.*

Si l'on se reporte au théorème de Laguerre démontré plus haut, on voit que ces courbes ne peuvent être que celles qui ont tous leurs foyers à l'infini; mais avant d'affirmer inversement que les courbes qui ont tous leurs foyers à l'infini jouissent de la propriété énoncée, on doit faire une discussion, très simple d'ailleurs.

Soit en effet G une courbe de classe n dont tous les foyers sont à l'infini: il est nécessaire pour cela que la droite de l'infini soit une tangente multiple d'ordre $n-1$, et que la courbe passe par les points cycliques du plan. Si ces conditions sont remplies, toutes les tangentes qu'on peut mener à G par les points cycliques coïncident avec la droite de l'infini, et tous les foyers de la courbe sont sur cette droite, mais leur position n'est pas déterminée, de sorte que le théorème de Laguerre ne paraît pas immédiatement applicable.

On peut voir néanmoins, d'une autre manière, que l'orientation du système des n tangentes menées à G d'un point quelconque 0 du plan, ne dépend pas de la position de ce point. Imaginons en effet que 0 décrive une droite; l'équation du système des n tangentes issues de 0 contient rationnellement un paramètre, et le théorème général du n° 3 est applicable. Par suite, l'orientation de ce système est fixe, si, toutes les fois qu'une ou plusieurs des tangentes issues de 0 passent par le point cyclique I, d'autres tangentes en même nombre passent par le point cyclique J. C'est précisément ce qui se présente ici; une tangente menée de 0 à G ne peut passer par I que si 0 est à l'infini, et elle passe alors par J. Nous pouvons donc énoncer ce théorème:

L'orientation du système des n tangentes menées d'un point à une courbe algébrique de classe n, $n-1$ fois tangente à la droite de l'infini et passant par les points cycliques, est indépendante de la position du point considéré dans le plan.

Les courbes dont il s'agit sont nécessairement tangentes à la droite de l'infini aux points cycliques, sinon on pourrait mener $n+1$ tangentes de l'un de ces points.

Il est facile de donner l'équation générale de ces courbes en coordonnées tangentielles rectangulaires; cette équation est:

$$(u^2+v^2)f_{n-2}(u, v) = F_n(u, v),$$

f étant un polynôme quelconque de degré $n-2$ en u et v, et F un polynôme également quelconque de degré n, mais homogène.

7. Parmi les courbes qui ont tous leurs foyers à l'infini, on peut citer, avec Laguerre, celles qui sont enveloppées par une droite dont deux points donnés décrivent respectivement deux courbes algébriques, quelconques d'ailleurs.

Un autre exemple intéressant est fourni par la famille des épicycloïdes algébriques.

Si une courbe a tous ses foyers à l'infini, c. à d. si les tangentes qu'on peut lui mener par les points cycliques coïncident toutes avec la droite de l'infini, la réciproque de cette courbe par rapport à un cercle de centre 0 sera telle que les droites isotropes issues de 0 ne la couperont qu'au point 0.

Or la réciproque d'une épicycloïde par rapport au cercle fixe a pour équation, en coordonnées polaires

$$\frac{1}{r} = k \cos \frac{1}{2n+1}\,\theta, \tag{3}$$

n désignant le rapport du rayon du cercle mobile à celui du cercle fixe; de plus n est positif pour l'épicycloïde, et négatif pour l'hypocycloïde.

M. Halphen a étudié d'une manière complète les points multiples des courbes (3), dont il écrit l'équation

$$\frac{1}{r} = k \cos \frac{p}{q}\,\theta, \tag{4}$$

p et q étant positifs et premiers entre eux. Il résulte de ses belles recherches que les courbes précédentes n'ont un point singulier à l'origine, 0, que si p est plus grand que q; en ce cas, la courbe présente en 0 deux *cycles*, dont les tangentes sont respectivement les droites isotropes; l'ordre de ces cycles est $p-q$ ou $\frac{p-q}{2}$, selon que p et q ne sont pas ou sont tous deux impairs; la classe des cycles est $2q$ ou q. D'ailleurs le degré de la courbe est $2p$ ou p. L'une des droites isotropes issues de 0 a en ce point avec la courbe un nombre d'intersections égal à la somme de l'ordre et de la classe du cycle correspondant, et de l'ordre de l'autre cycle, c. à d. égal à $2p$ ou à p, ou, si l'on veut, égal dans tous les cas au degré de la courbe. Elle ne coupe donc la courbe qu'au point 0.

Il résulte de là qu'une épicycloïde ou hypocycloïde algébrique aura tous ses foyers à l'infini si sa réciproque a une équation de la forme (4), p et q étant positifs, premiers entre eux, et p étant supérieur à q.

Soit alors $n = \frac{\lambda}{\mu}$, λ et μ étant premiers entre eux; on aura

$$\frac{p}{q} = \pm \frac{1}{2n+1} = \pm \frac{\mu}{2\lambda + \mu},$$

si μ et λ sont positifs, c. à d. si la courbe est une épicycloïde, p sera toujours inférieur à q.

Si la courbe est une hypocycloïde, on devra supposer λ négatif, et l'on aura, si $\lambda = -\lambda'$:

$$\frac{p}{q} = \pm \frac{\mu}{\mu - 2\lambda'}.$$

Il faut, pour que p soit supérieur à q, que μ soit, en valeur absolue, supérieur à $\mu - 2\lambda'$, c. à d. que λ' soit plus petit que μ; n est alors, en valeur absolue, inférieur à 1. Donc:

Les hypocycloïdes algébriques obtenues en faisant rouler un cercle à l'intérieur d'un cercle plus grand ont tous leurs foyers à l'infini; et par suite *l'orientation du système des tangentes menées d'un point du plan à l'une de ces courbes est indépendante de la position du point.*

Les autres courbes de la famille des épicycloïdes ou hypocycloïdes algébriques ne possèdent pas la même propriété.

III.—*Application à l'hypocycloïde à trois rebroussements.*

8. La plus simple des hypocycloïdes qu'on vient de rencontrer est, après la droite qui correspond au cas de $n = -\frac{1}{2}$, l'hypocycloïde à trois rebroussements, qui correspond à celui de $n = -\frac{1}{3}$; le théorême précédent donne une propriété des tangentes à cette courbe qui paraît nouvelle, et qu'on peut énoncer ainsi:

L'orientation du système des trois tangentes menées d'un point quelconque à une hypocycloïde à trois rebroussements est la même que celle des trois axes de symétrie de la courbe.

Ce théorême permet, lorsqu'on connaît deux tangentes de l'hypocycloïde, de construire immédiatement et sans ambiguïté la troisième tangente qu'on peut mener par le point d'intersection des deux premières. On peut le regarder comme l'interprétation géométrique, dans le cas de l'hypocycloïde, de la propriété analytique fondamentale des courbes de troisième classe, propriété bien connue qu'on peut énoncer ainsi: il est possible de faire correspondre à chaque tangente d'une courbe de troisième classe un argument, de telle sorte que les arguments des trois tangentes issues d'un point quelconque aient une somme constante. En général, on ne connaît pas la signification *géométrique* de ces arguments, qui s'introduisent par la considération des fonctions elliptiques ; dans le cas de l'hypocycloïde, on voit que cette signification est très simple, l'argument étant l'angle que fait la tangente avec un des axes de symétrie de la courbe.

Il importe, pour ce qui va suivre, de préciser cette notion : soit t une tangente de l'hypocycloïde ; si par un point fixe 0 nous menons une parallèle à t et une parallèle $0x$ à l'un des axes de la courbe, choisi une fois pour toutes, nous désignerons par α l'angle que font ces deux droites, en le comptant à partir de $0x$, dans le sens trigonométrique. Cet angle n'est défini qu'à un multiple près de π, cequi n'a aucun inconvénient, puisque les orientations sont définies dans les mêmes conditions.

9. Cela posé, on déduit aisément du théorême fondamental les conséquences suivantes.

Soit t une tangente en un point A de l'hypocycloïde : les bissectrices de l'angle des deux tangentes, autres que t, que l'on peut mener à la courbe par un point de cette droite, sont parallèles à deux directions fixes.

Ces directions sont celles des tangentes aux points B et C, où la tangente t rencontre de nouveau l'hypocycloïde.

On voit ainsi qu'une tangente à la courbe la rencontre de nouveau en deux points, où les tangentes sont perpendiculaires l'une à l'autre : proposition bien connue, qui sert de base au beau mémoire de M. Cremona sur l'hypocycloïde.

Par deux points quelconques d'une tangente t à l'hypocycloïde menons à la courbe les quatre tangentes autres que t : ces quatre droites forment un quadrilatère inscriptible dans un cercle.

Réciproquement, si le quadrilatère complet formé par quatre tangentes de l'hypocycloïde a quatre de ses sommets sur un cercle, la droite qui joint les deux autres sommets est une tangente de la courbe.

Dans tout triangle isocèle circonscrit à une hypocycloïde, la droite qui joint le sommet au point de contact de la base est une tangente de la courbe.

Par chaque sommet d'un triangle circonscrit à une hypocycloïde passe une nouvelle tangente, distincte des côtés du triangle : les trois droites ainsi définies forment un nouveau triangle semblable au premier.

10. Ce dernier théorème mérite d'être étudié avec quelques détails; il donne lieu à des conséquences intéressantes.

D'un triangle ABC, circonscrit à l'hypocycloïde, on déduit, en menant les tangentes, distinctes des côtés, qui passent par les trois sommets, un nouveau triangle $A_1B_1C_1$ semblable au premier; en appliquant la même construction à $A_1B_1C_1$, on obtient un troisième triangle semblable aux deux premiers, et ainsi de suite. Cette série de triangles est-elle illimitée? Retombe-t-on nécessairement sur un des triangles déjà trouvés, ou l'un des triangles finit-il par se réduire à un point? Ce sont là des questions auxquelles il est facile de répondre par l'application du théorème fondamental.

Désignons par α, β, γ les angles que font avec un des axes de symétrie de la courbe les côtés du triangle ABC; soit γ_1 l'angle que fait avec ce même axe la troisième tangente issue du point C. On a, d'après le théorème fondamental:

$$\gamma_1 + \alpha + \beta = h\pi,$$

d'où

$$\gamma_1 \equiv \gamma - (\alpha + \beta + \gamma) \mod \pi.$$

Par suite les angles α_1, β_1, γ_1, que font avec l'axe considéré les trois côtés du triangle $A_1B_1C_1$ sont:

$$\begin{aligned} \alpha_1 &\equiv \alpha - (\alpha + \beta + \gamma) \mod \pi, \\ \beta_1 &\equiv \beta - (\alpha + \beta + \gamma), \\ \gamma_1 &\equiv \gamma - (\alpha + \beta + \gamma). \end{aligned}$$

Ce sont ces relations qui montrent la similitude des triangles ABC et $A_1B_1C_1$, puisque l'on en tire évidemment

$$\alpha_1 - \beta_1 = \alpha - \beta, \quad \alpha_1 - \gamma_1 = \alpha - \gamma, \quad \beta_1 - \gamma_1 = \beta - \gamma.$$

Rien n'est plus aisé que de déterminer le rapport de similitude.

Remarquons en effet que les côtés homologues des deux triangles se coupent sous des angles égaux à $(\alpha + \beta + \gamma)$; or, d'après un théorème connu de géométrie élémentaire, si, par les sommets d'un triangle, on mène des droites faisant avec les côtés opposés, dans un même sens de rotation, des angles égaux ω, on forme avec ces droites un nouveau triangle semblable au premier, avec un rapport de similitude égal à $2 \cos \omega$.

Les triangles $A_1B_1C_1$ et ABC sont donc semblables avec un rapport de similitude égal à $2 \cos (\alpha + \beta + \gamma)$.

11. De là se déduisent de suite quelques résultats simples.

En premier lieu, si $\alpha+\beta+\gamma\equiv\pm\frac{\pi}{3}$, $\cos(\alpha+\beta+\gamma)$ est égal à $\frac{1}{2}$, et les triangles ABC, $A_1B_1C_1$ sont égaux. On a d'ailleurs

$$\alpha_1+\beta_1+\gamma_1\equiv-2(\alpha+\beta+\gamma)\equiv\pm\frac{\pi}{3}\mod\pi$$

et le triangle $A_2B_2C_2$ déduit de $A_1B_1C_1$ sera égal aux deux précédents. On a pour ce triangle:

$$\alpha_2\equiv\alpha_1\mp\frac{\pi}{3}\equiv\alpha\pm\frac{\pi}{3}\mod\pi.$$

. . . .

De même pour le triangle $A_3B_3C_3$ déduit de $A_2B_2C_2$, on aura:

$$\alpha_3\equiv\alpha_2-(\alpha_2+\beta_2+\gamma_2)\equiv\alpha\mod\pi.$$

. . . .

Le triangle $A_3B_3C_3$ coïncide donc avec ABC, puisqu'on ne peut mener à l'hypocycloïde qu'une seule tangente parallèle à une droite donnée.

On peut donc énoncer la proposition suivante:

Pour simplifier, appelons *premier triangle dérivé*, ou, plus simplement, *triangle dérivé* d'un triangle T, circonscrit à l'hypocycloïde, le triangle T_1 formé par les tangentes menées à la courbe des sommets de T, et distinctes des côtés de T; appelons *second triangle dérivé* de T le premier triangle dérivé de T_1, et ainsi de suite. On a en premier lieu la proposition générale:

Tous les triangles dérivés d'un même triangle lui sont semblables. La propriété démontrée plus haut dans le cas où $\alpha+\beta+\gamma=\pm\frac{\pi}{3}$ s'énonce ainsi:

Si les côtés d'un triangle T, circonscrit à l'hypocycloïde, font avec un des axes de symétrie de la courbe des angles dont la somme est $\pm\frac{\pi}{3}$, à un multiple près de π, les deux premiers triangles dérivés de T sont égaux à ce triangle, et le troisième coïncide avec T.

En second lieu, si $\alpha+\beta+\gamma=\frac{\pi}{2}$, le rapport de similitude est nul; le triangle $A_1B_1C_1$ se réduit donc à un point. De plus on a:

$$\alpha_1=\alpha-\frac{\pi}{2}\mod\pi,$$

ce qui montre que les côtés de $A_1B_1C_1$ sont perpendiculaires à ceux de ABC. Donc:

Si les côtés d'un triangle circonscrit à l'hypocycloïde, font avec un des axes de symétrie de la courbe des angles dont la somme est $\frac{\pi}{2}$, *à un multiple près de* π, *les hauteurs de ce triangle sont des tangentes de l'hypocycloïde, et le triangle dérivé se réduit par suite à un point.*

12. Reprenons maintenant les relations

$$\alpha_1 \equiv \alpha - (\alpha + \beta + \gamma) \mod \pi,$$
$$\dots\dots\dots\dots\dots$$

entre les angles qui correspondent à un triangle ABC et au triangle dérivé $A_1B_1C_1$. On a de même, en passant au dérivé de $A_1B_1C_1$:

$$\alpha_2 \equiv \alpha_1 - (\alpha_1 + \beta_1 + \gamma_1) \equiv \alpha + (\alpha + \beta + \gamma) \mod \pi,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

En général, pour le $n^{\text{ième}}$ triangle dérivé de ABC, on aura des expressions de la forme

$$\alpha_n \equiv \alpha + h_n(\alpha + \beta + \gamma) \mod \pi,$$
$$\beta_n = \beta + h_n(\alpha + \beta + \gamma),$$
$$\gamma_n = \gamma + h_n(\alpha + \beta + \gamma).$$

Pour le $(n+1)^{\text{ième}}$ triangle, il viendra:

$$\alpha_{n+1} \equiv \alpha_n - (\alpha_n + \beta_n + \gamma_n) \equiv \alpha - [2h_n + 1](\alpha + \beta + \gamma),$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

d'où, la loi de récurrence:

$$h_{n+1} + 2h_n + 1 = 0.$$

On en tire, puisque $h_1 = -1$:

$$h_n = \frac{(-2)^n - 1}{3},$$

et par suite, les angles α_n, β_n, γ_n que font avec l'axe les côtés du $n^{\text{ième}}$ triangle dérivé de ABC sont donnés, en fonction des angles analogues α, β, γ qui correspondent à ce triangle, par les formules:

$$\alpha_n = \alpha + \frac{(-2)^n - 1}{3}(\alpha + \beta + \gamma),$$
$$\beta_n = \beta + \frac{(-2)^n - 1}{3}(\alpha + \beta + \gamma),$$
$$\gamma_n = \gamma + \frac{(-2)^n - 1}{3}(\alpha + \beta + \gamma).$$

Observons enfin que le rapport de similitude des triangles

$$A_nB_nC_n \text{ et } A_{n-1}B_{n-1}C_{n-1}$$

est égal, d'après un résultat rappelé plus haut, à

$$2\cos\frac{(-2)^n-(-2)^{n-1}}{3}(\alpha+\beta+\gamma),$$

c. à d. à $$2\cos 2^{n-1}(\alpha+\beta+\gamma).$$

13. Ces formules permettent de répondre aux questions que l'on s'était posées.

D'abord, dans quels cas la suite des triangles dérivés l'un de l'autre se terminera-t-elle à un point?

Pour que le triangle $A_nB_nC_n$ se réduire à un point, il faut que le rapport de similitude de ce triangle avec le triangle $A_{n-1}B_{n-1}C_{n-1}$ soit nul, c. à d. que:

$$2^{n-1}(\alpha+\beta+\gamma)\equiv\frac{\pi}{2}\mod\pi,$$

ou $$\alpha+\beta+\gamma=\frac{2k+1}{2^n}\pi.$$

On aurait pu arriver de suite à ce résultat en écrivant que $\alpha_n+\beta_n+\gamma_n$ est nul à un multiple près de π.

De là ce théorême, qui est la généralisation d'un résultat donné plus haut.

Si les côtés d'un triangle circonscrit à l'hypocycloïde font avec un des axes de symétrie de la courbe des angles dont la somme est de la forme $\frac{2k+1}{2^n}\pi$, *la série des triangles dérivés du premier se terminera à un point, au bout de n constructions.*

Ce point sera le point de concours des hauteurs du $(n-1)^{\text{ième}}$ triangle dérivé du triangle primitif.

Cherchons maintenant dans quels cas on retombera, après un certain nombre de constructions sur le triangle primitif.

Il faut pour cela que l'on ait $\alpha_n\equiv\alpha$, $\beta_n\equiv\beta$, $\gamma_n\equiv\gamma \mod \pi$, c. à d.:

$$\alpha+\beta+\gamma=\frac{3k}{(-2)^n-1}\pi.$$

Si n est le plus petit nombre pour lequel une relation de cette forme ait lieu, le $n^{\text{ième}}$ triangle dérivé du triangle primitif coïncidera avec ce triangle, et si l'on continue les constructions, on retrouve tous les triangles déjà formés.

Mais ici se présente une particularité curieuse: c'est qu'en construisant les triangles successifs à partir du premier, il peut arriver que l'un d'eux coïncide avec l'un des précédents sans que l'on ait retrouvé de nouveau le premier triangle.

En effet, le $p^{\text{ième}}$ et le $q^{\text{ième}}$ triangles dérivés ($q > p$) coïncideront si l'on a:

$$\frac{(-2)^p-1}{3}(\alpha+\beta+\gamma)=\frac{(-2)^q-1}{3}(\alpha+\beta+\gamma)+k\pi,$$

c. à d.:

$$(-2)^p(\alpha+\beta+\gamma)[(-2)^{q-p}-1]=3k\pi,$$

ou

$$\alpha+\beta+\gamma=\frac{3k}{(-2)^p[(-2)^{q-p}-1]}\pi.$$

Si cette condition est remplie, k étant premier à 2, le $q^{\text{ième}}$ triangle dérivé coïncidera avec le $p^{\text{ième}}$, et, en continuant les constructions, on retrouvera indéfiniment les triangles dérivés dont l'ordre est compris entre p et $q-1$, sans retomber jamais sur le triangle primitif et les $p-1$ premiers triangles dérivés. Ainsi:

Si les côtés d'un triangle circonscrit à l'hypocycloïde font avec un des axes de symétrie de la courbe des angles dont la somme est de la forme $\frac{3k}{2^p[(-2)^{q-p}-1]}\pi$, *le* $q^{\text{ième}}$ *triangle dérivé de ce triangle coïncidera avec le* $p^{\text{ième}}$.

14. Il est aisé d'expliquer à priori pourquoi, dans le cas qui nous occupe, le triangle primitif ne se reproduit pas nécessairement: cela tient à ce que la suite des triangles dérivés n'est pas réversible sans ambiguité, ou, en termes plus précis, à ce qu'un même triangle circonscrit à l'hypocycloïde peut être considéré comme le dérivé de deux autres triangles circonscrits, et non pas d'un seul.

Reprenons en effet les relations

$$\alpha_1=\alpha-(\alpha+\beta+\gamma)+k\pi,$$
$$\beta_1=\beta-(\alpha+\beta+\gamma)+l\pi,$$
$$\gamma_1=\gamma-(\alpha+\beta+\gamma)+m\pi.$$

On en tire:

$$\alpha=\alpha_1-\frac{1}{2}(\alpha_1+\beta_1+\gamma_1)+\frac{l+m-k}{2}\pi,$$
$$\beta=\beta_1-\frac{1}{2}(\alpha_1+\beta_1+\gamma_1)+\frac{k+m-l}{2}\pi,$$
$$\gamma=\gamma_1-\frac{1}{2}(\alpha_1+\beta_1+\gamma_1)+\frac{k+l-m}{2}\pi.$$

Les nombres $l+m-k$, $k+m-l$, $k+l-m$ sont de même parité; on aura donc, pour α, β, γ, à des multiples de π près, les deux systèmes de valeurs:

$$\begin{aligned} \alpha &= \alpha_1 - \frac{1}{2}(\alpha_1+\beta_1+\gamma_1), & \alpha &= \alpha_1 - \frac{1}{2}(\alpha_1+\beta_1+\gamma_1) + \frac{\pi}{2}, \\ \beta &= \beta_1 - \frac{1}{2}(\alpha_1+\beta_1+\gamma_1), & \beta &= \beta_1 - \frac{1}{2}(\alpha_1+\beta_1+\gamma_1) + \frac{\pi}{2}, \\ \gamma &= \gamma_1 - \frac{1}{2}(\alpha_1+\beta_1+\gamma_1), & \gamma &= \gamma_1 - \frac{1}{2}(\alpha_1+\beta_1+\gamma_1) + \frac{\pi}{2}; \end{aligned}$$

De là ce théorème:

Dans tout triangle circonscrit à une hypocycloïde, on peut inscrire deux autres triangles circonscrits à la courbe: ces deux triangles sont semblables au premier et semblables entre eux; leurs côtés homologues sont rectangulaires.

15. On pourrait pousser plus loin ces recherches, en étudiant l'ensemble des triangles circonscrits qui admettent pour triangle dérivé d'un ordre donné un même triangle circonscrit, et l'on arriverait ainsi à des résultats assez curieux que nous n'énoncerons pas, afin de ne pas fatiguer l'attention du lecteur; nous nous contenterons de signaler une proposition de nature différente qui se déduit aisément du théorème fondamental.

Soient A, B, C les points de contact des tangentes menées à l'hypocycloïde par un point M; sur les directions MA, MB, MC portons, à partir de M, des longueurs Ma, Mb, Mc, respectivement égales aux inverses des segments MA, MB, MC: le point M est le centre de gravité du triangle abc.

On peut donner une autre forme à ce théorème:

La tangente en M au cercle qui passe par les points M, A, B, est conjuguée harmonique de MC par rapport aux droites MA et MB; de plus, si l'on porte sur la droite MC, à partir de M et dans le sens opposé à MC, une longueur égale à $2MC$, l'extrémité du segment ainsi obtenu est sur le cercle précédent.

Sans insister sur les conséquences que l'on pourrait déduire de ces propositions pour la théorie de l'hypocycloïde à trois rebroussements, nous reviendrons au théorème démontré plus haut, relativement au lieu des foyers d'un faisceau tangentiel de courbes planes, et nous montrerons qu'il met en évidence une classe intérressante de courbes, étudiées déjà par divers géomètres, et en particulier par M. Darboux.

IV.—*Lieu des foyers d'un faisceau tangentiel de courbes planes.*

16. Nous avons démontré plus haut que :

Le lieu des foyers des courbes d'un faisceau tangentiel déterminé par deux courbes A et B, de classe n, est une courbe F, telle que si l'on joint un de ses points aux n foyers réels de A et aux n foyers réels de B, les deux systèmes de droites ainsi obtenus aient même orientation.

Ce résultat peut être présenté sous une autre forme. Groupons deux à deux, d'une manière quelconque, un foyer a_κ de la courbe A et un foyer b_κ de la courbe B; il est clair que le lieu F est celui des points tels que, de l'un quelconque d'entre eux, les n segments $a_\kappa b_\kappa$ soient vus sous des angles ayant une somme algébrique égale à un multiple de π, et l'on retrouve ainsi des courbes remarquables étudiées par M. Darboux.

Ces courbes comprennent, comme cas particuliers, les courbes telles que l'on voie, de chacun de leurs points, n segments fixes sous des angles dont la somme algébrique est égale à une constante quelconque: il suffit en effet de supposer que chacune des courbes A et B a un foyer à l'infini, c. à d. que ces courbes touchent toutes deux la droite de l'infini; un des segments $a_\kappa b_\kappa$ est alors à l'infini, et il est vu de tout point du plan sous un angle constant, θ. La somme des angles sous lesquels les autres segments à distance finie sont vus d'un point quelconque du lieu F est donc constante, et égale à $-\theta$, à un multiple près de π.

De là définition même du lieu F résulte immédiatement une belle proposition, donnée par M. Darboux :

Si une courbe est telle que, de chacun de ses points, plusieurs segments soient vus sous des angles dont la somme est un multiple de π, elle conserve la même propriété avec une infinité d'autres segments ayant tous leurs extrémités sur la courbe.

Ces segments s'obtiennent en joignant deux à deux, d'une manière quelconque, les foyers de deux courbes quelconques du faisceau tangentiel déterminé par les courbes A et B qui ont servi à la définition primitive du lieu.

Si A et B touchent la droite de l'infini, il résulte de ce qui à été dit plus haut que la proposition précédente doit être modifiée ainsi :

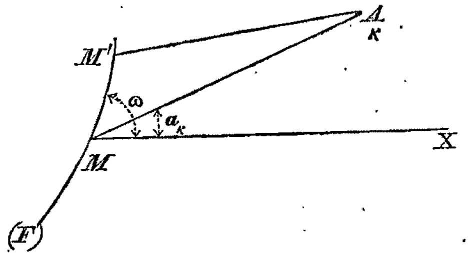
Si une courbe est telle que, de chacun de ses points, plusieurs segments soient vus sous des angles dont la somme est constante, elle conserve la même propriété avec une infinité d'autres segments, mais la valeur de la somme constante varie quand on passe d'un système de segments à l'autre.

La définition du lieu F ne dépend que de la position des foyers des courbes A et B; on peut en particulier supposer que chacune de ces courbes se réduise à ses n foyers réels, et l'on a ce théorème:

Si une courbe est telle qu'en joignant un quelconque de ses points à deux séries de n pôles fixes, à distance finie ou infinie, on obtienne deux systèmes de même orientation, cette courbe est le lieu des foyers des courbes de classe n qui touchent les n^2 droites joignant les pôles de l'une des séries aux pôles de l'autre série.

17. L'ensemble des n foyers réels d'une courbe appartenant à un faisceau tangentiel donné jouit de quelques propriétés simples, qui dérivent aisément des principes précédents.

Soient en effet $A_1, A_2, \ldots A_n$ les n foyers réels de la courbe A; $B_1 \ldots B_n$ ceux de la courbe B; M le foyer d'une courbe du faisceau tangentiel déterminé par A et B; M' le point infiniment voisin de M sur le lieu F.



Menons par M un axe quelconque MX; désignons par ω l'angle $M'MX$, par α_κ l'angle $A_\kappa MX$; par β_κ l'angle $B_\kappa MX$. On a, d'après la propriété fondamentale du lieu F:

$$\alpha_1 + \alpha_2 + \ldots + \alpha_n = \beta_1 + \beta_2 + \ldots + \beta_n$$

et, en passant de M à M',

$$d\alpha_1 + d\alpha_2 + \ldots = d\beta_1 + d\beta_2 + \ldots$$

Or $d\alpha_\kappa$ est l'angle $M'A_\kappa M$, et l'on a, dans le triangle $M'A_\kappa M$:

$$d\alpha_\kappa = \frac{MM'}{MA_\kappa} \sin(\omega - \alpha_\kappa).$$

Par suite:

$$\sum \frac{\sin(\omega - \alpha_\kappa)}{MA_\kappa} = \sum \frac{\sin(\omega - \beta_\kappa)}{MB_\kappa}. \tag{5}$$

18. Ce résultat est susceptible d'une interprétation géométrique élégante, si l'on introduit une notion due à Laguerre.

Cette notion est celle du *centre harmonique* d'un système de points par rapport à un point.

Etant donnés dans un plan un point M et un groupe de n points, $A_1 \ldots . A_n$, portons sur chaque droite MA_κ, à partir de M, une longueur égale à l'inverse de MA_κ: composons ces longueurs comme des forces; sur la direction de leur résultante, portons, à partir de M, une longueur égale à l'inverse de la $n^{\text{ième}}$ partie de cette résultante; l'extrémité, a, du segment obtenu sera dit le centre harmonique des points $A_1 \ldots . A_n$ relativement au point M.

D'après cette définition, si l'on imagine par M deux axes rectangulaires, MX et MY, et si α_κ est l'angle de MA_κ avec MX, les coordonnées du centre harmonique a seront:

$$X = n\frac{\xi}{\xi^2+\eta^2}, \qquad Y = n\frac{\eta}{\xi^2+\eta^2},$$

étant posé

$$\xi = \Sigma\frac{\cos\alpha_\kappa}{MA_\kappa}, \qquad \eta = \Sigma\frac{\sin\alpha_\kappa}{MA_\kappa}.$$

On peut aussi écrire:

$$\xi = n\frac{X}{X^2+Y^2}, \qquad \eta = n\frac{Y}{X^2+Y^2}.$$

On aura des formules analogues pour les coordonnées des centre harmonique, b, des points $B_1 \ldots . B_n$ par rapport à M:

$$X' = n\frac{\xi'}{\xi'^2+\eta'^2}, \qquad Y' = n\frac{\eta'}{\xi'^2+\eta'^2},$$

étant posé

$$\xi' = \Sigma\frac{\cos\beta_\kappa}{MB_\kappa}, \qquad \eta' = \Sigma\frac{\sin\beta_\kappa}{MB_\kappa}.$$

Or la relation (5) donne:

$$\xi\sin\omega - \eta\cos\omega = \xi'\sin\omega - \eta'\cos\omega,$$

remplaçons dans cette équation ξ et η, ξ' et η' par leurs valeurs en X et Y, X' et Y', il vient:

$$\frac{X\sin\omega - Y\cos\omega}{X^2+Y^2} = \frac{X'\sin\omega - Y'\cos\omega}{X'^2+Y'^2}.$$

En d'autres termes, un même cercle:

$$x^2+y^2+\lambda(x\sin\omega - y\cos\omega) = 0$$

passe par a et b: ce cercle est d'ailleurs tangent en M à la direction MM', c. à d. à la courbe F. Donc, les centres harmoniques des points $A_1 \ldots\ldots A_n$ et $B_1 \ldots\ldots B_n$ par rapport à un même point M du lieu F, sont sur un cercle tangent à F au point M, et, comme on peut remplacer les points $A_1 \ldots\ldots A_n$, ou $B_1 \ldots\ldots B_n$ par les n foyers réels d'une quelconque des courbes du faisceau tangentiel determiné par les courbes A et B, on a ce résultat:

Soit F le lieu des foyers des courbes de classe n appartenant à un faisceau tangentiel donné: le centre harmonique des n foyers réels de l'une quelconque de ces courbes par rapport à un point choisi arbitrairement sur F reste sur un cercle, tangent en ce point à la courbe F.

19. Un cas particulier remarquable est celui où le point M est un *point double* de la courbe F; l'équation

$$\xi \sin\omega - \eta\cos\omega = \xi' \sin\omega - \eta' \cos\omega$$

est alors vérifiée pour les deux valeurs de ω qui correspondent aux deux branches de la courbe passant en M, et, par suite, on a nécessairement $\xi = \xi'$, $\eta = \eta'$. Donc:

Si la courbe F a un point double, le centre harmonique des n foyers réels d'une quelconque des courbes du faisceau tangentiel par rapport à ce point, est un point fixe.

Si M est à l'infini, et ne coïncide pas avec un des points cycliques du plan, il est aisé de voir que le centre harmonique d'un groupe de points par rapport à M coïncide avec le centre des moyennes distances de ces points; or on prouve facilement que, si aucune des courbes A et B n'a de foyer à l'infini, la courbe F a une asymptote réelle, et par suite, il résulte du théorême démontré plus haut, que le centre des moyennes distances des n foyers réels d'un courbe variable, appartenant à un faisceau tangentiel déterminé, décrit une *droite*.

L'application de ces théorêmes généraux au cas d'un faisceau tangentiel de coniques présente quelque intérêt.

V.—*Lieu des foyers d'un faisceau tangentiel de coniques.*

20. Soient deux coniques, A et B, ayant respectivement pour foyers réels les points f et f', g et g': le lieu, F, des foyers des coniques inscrites dans le

même quadrilatère que les coniques A et B est, d'après la théorie générale, le lieu des points M tels que les systèmes de droites Mf et Mf', Mg et Mg' aient mêmes bissectrices.

Rien n'est plus facile que d'obtenir, en partant de là, l'équation du lieu: on trouverait une cubique, passant par les points cycliques du plan, et ayant une asymptote parallèle à la droite qui joint les milieux des segments ff' et gg', c. à d. les centres des coniques A et B.

Cette cubique peut être déterminée par points d'une manière très simple.

La conique A a deux foyers imaginaires, qui s'obtiennent en joignant f et f' aux points cycliques I et J, et en prenant les intersections des droites ainsi obtenues; ces deux nouveaux points, f_1 et f_1', sont aussi sur le lieu des foyers F.

De même, si l'on joint les points f et f' aux points g et g', les droites fg et $f'g'$, fg' et $f'g$ se coupent respectivement en deux nouveaux points, k et k', qui sont sur F, d'après la propriété fondamentale de ce lieu.

Il y a plus: les couples de points f_1 et f_1', k et k' jouent le même rôle que les couples f et f' ou g et g'. La proposition est évidente pour les points f_1 et f_1' qui sont des foyers d'une des coniques du faisceau; on peut montrer de même que k et k' sont aussi les foyers d'une de ces coniques: en effet, une conique du faisceau a un foyer en k, puisque k est sur F; le second foyer réel de cette conique est nécessairement en k', puisque les couples de droites fk, fk' et $f'k$, $f'k'$ ont mêmes bissectrices que les couples fg, fg' et $f'g$, $f'g'$, avec lesquels ils coïncident.

En d'autres termes, le lieu F peut être défini, au moyen des 3 couples de points f et f', g et g', I et J, de la manière suivante: on joint deux à deux les points de deux de ces couples, on obtient, par les intersections des droites ainsi construites, un nouveau couple; en opérant de la même manière sur ce nouveau couple et sur le troisième, on obtient un cinquième couple; et on continue ainsi indéfiniment en combinant deux quelconques des couples obtenus; tous ces couples sont sur F. On reconnaît la construction discontinue donnée par Schröter pour les courbes du troisième ordre; les couples de points considérés sont des couples *de pôles conjugués;* ils jouissent de la propriété que les tangentes menées à la cubique aux deux points d'un couple se rencontrent sur la courbe.

21. On peut dire d'après cela que *le lieu des foyers des coniques inscrites dans un quadrilatère est une cubique circulaire, dont les tangentes aux points cycliques se*

coupent sur la courbe; ou, plus simplement, une cubique circulaire qui passe par son foyer singulier.

Réciproquement, toute cubique circulaire passant par son foyer singulier peut être considérée comme le lieu des foyers de coniques inscrites dans un quadrilatère: il suffit de prendre sur cette cubique deux couples de pôles conjugués quelconques, f et f', g et g', du même système que le couple formé par les points cycliques, et la cubique est le lieu des foyers des coniques du faisceau tangentiel déterminé par deux coniques, quelconques d'ailleurs, ayant respectivement pour foyers les points f et f', g et g'.

La propriété caractéristique du lieu F peut, par une transformation homographique, être mise sous la forme suivante:

Les deux couples de droites qui joignent un point quelconque d'une cubique à deux couples de pôles conjugués d'un même système, situés sur cette cubique, sont en involution.

Ce théorème est dû à M. Cremona.

22. Les propositions démontrées plus haut relativement aux centres harmoniques prennent une forme assez simple si l'on observe, avec Laguerre, que le centre harmonique, a, de deux points, f et f', par rapport à un point M est sur la circonférence passant par M, f et f', et que les points M, a, f, f' divisent harmoniquement cette circonférence.

Par un point M, du lieu F des coniques inscrites dans un quadrilatère, et les foyers f et f' de l'une quelconque de ces coniques, faisons passer un cercle, et prenons, sur ce cercle, le point a, qui, avec les points M, f, f' divise harmoniquement la circonférence: d'après un théorème établi plus haut, *le point a décrit un cercle tangent en M à la courbe F.*

On voit aisément que ce cercle passe par le point obtenu en prolongeant d'une longueur égale à elle-même la droite qui va de M au foyer singulier de la cubique.

Si la courbe F a un point double, le centre harmonique des deux points d'un même couple, f et f', par rapport à ce point double est un point fixe: il est très facile d'établir que la courbe F n'aura de point double que si le faisceau de coniques qui sert à la définir contient un cercle; le point double est alors le foyer du cercle. Donc:

Si, par les foyers f et f' de l'une quelconque des coniques inscrites dans un quadrilatère circonscrit à un cercle de centre o, et par le point o on fait passer une circonférence, cette circonférence passe par un second point fixe, o', qui, avec les points o, f et f', la divise harmoniquement. Le segment oo' a pour milieu le foyer singulier de la cubique lieu des foyers f, f'.

La courbe F est alors ce que Quételet appelle une *focale à nœud;* une focale à nœud est, d'après ce qui précède, une cubique unicursale passant par les points cycliques et par son foyer singulier.

23. On rencontre cette courbe dans un problème assez intéressant de la Géométrie de l'espace.

Chasles a fait voir que le lieu des pieds des normales à un système de quadriques homofocales, contenues dans un plan donné P, est une focale à nœud, et nous avons démontré que le lieu des foyers des sections faites par le plan P dans les quadriques homofocales coïncide avec le lieu de Chasles. On peut établir directement que le lieu des foyers est une focale à nœud, en s'appuyant sur les principes généraux énoncés au commencement de ce travail, et en déduire quelques propriétés géométriques simples.

24. Cherchons en effet l'équation tangentielle, dans le plan P, des coniques communes à ce plan et aux surfaces homofocales.

Il y a deux quadriques du système homofocal qui touchent une droite donnée; donc deux des coniques, dans le plan P, touchent une droite de ce plan, et l'équation générale cherchée contiendra un paramètre, θ, au second degré. Elle sera de la forme:

$$A\theta^2 + B\theta + C = 0,$$

et l'on pourra supposer que les coniques $A = 0$ et $C = 0$ sont deux quelconques des coniques du système.

Or parmi ces coniques figure celle qui se compose des deux points cycliques du plan P, puisque le cercle à l'infini fait partie du système homofocal; on peut donc supposer que l'on a:

$$A = u^2 + v^2,$$

et l'équation devient

$$(u^2 + v^2)\theta^2 + B\theta + C = 0.$$

Les coniques ainsi représentées sont homofocales aux coniques

$$B\theta + C = 0,$$

qui appartiennent à un même faisceau tangentiel.

Cette remarque suffit pour établir que le lieu des foyers des coniques est une cubique circulaire, passant par son foyer singulier ; observons maintenant qu'une des quadriques du système homofocal touche le plan P, et par suite une des coniques se réduit (au point de vue tangentiel) au point de contact compté deux fois ; ce point est donc un point double de la cubique, qui est dès lors une focale à nœud.

Si maintenant on désigne par f et f', g et g', les foyers de deux des coniques, la focale à nœud peut être considérée, comme le lieu des foyers des coniques tangentes aux droites $fg, fg', f'g, f'g'$, et, puisqu'elle a un point double, ces quatre droites doivent nécessairement toucher un cercle décrit du point double comme centre.

25. On peut donc énoncer les propositions suivantes.

Le lieu des foyers des sections faites dans une série de quadriques homofocales par un plan P est une focale à nœud, dont le point double est le point de contact, o, du plan P avec la quadrique du système qui touche ce plan.

Si par le point o et les foyers f et f' de l'une des coniques d'intersection on fait passer un cercle, ce cercle passe par un second point fixe, o', qui, avec les points o, f et f' divise harmoniquement la circonférence.

Le segment oo' a pour milieu le foyer singulier, ϕ, de la focale ; ce point ϕ est le foyer de la parabole qui figure parmi les coniques d'intersection.

Les quatre droites qui joignent les foyers réels de l'une des coniques d'intersection aux foyers réels d'une autre de ces coniques touchent un même cercle qui a pour centre le point o.

En particulier :

Les droites qui joignent deux à deux les points d'intersection de deux coniques, focales l'une de l'autre, par un même plan, forment un quadrilatère circonscriptible à un cercle.

VI.—*Propriété des foyers des courbes appartenant à un faisceau tangentiel.*

26. Nous avons démontré au n° 18, comme conséquence d'une proposition fondamentale, que le centre harmonique des n foyers réels d'une courbe appartenant à un faisceau tangentiel de classe n, par rapport à un point du lieu F correspondant, reste sur un cercle, tangent en ce point à la courbe F.

Par des considérations d'un autre ordre, sur lesquelles nous aurons à revenir, à un point de vue plus général, dans un travail ultérieur, on arrive à un résultat qui complète le précédent, et que nous nous bornerons à énoncer :

Le centre harmonique, par rapport à un point du plan, des foyers réels de chacune des courbes d'un faisceau tangentiel, décrit un cercle.

Ce théorême peut recevoir une forme plus élégante si l'on observe avec Laguerre que le centre harmonique des foyers réels d'une courbe par rapport à un point coïncide avec le centre harmonique des points de contact des tangentes qu'on peut mener de ce point à la courbe. Ainsi :

Le centre harmonique, par rapport à un point du plan, des points de contact des tangentes qu'on peut mener de ce point à chacune des courbes d'un même faisceau tangentiel décrit un cercle.

THE BENEKE PHILOSOPHICAL PRIZE.

The Philosophical Faculty of the University of Göttingen proposes the following subject for discussion by the competitors for the above named prize for the year 1891:

Of late years it has become more and more evident that the law of Entropy is of fundamental importance for the theory of all those physical and chemical phenomena which are connected with the production or absorption of heat. In the treatises on the law of energy written for the Beneke prize of 1884, it was especially recognized that the law of energy requires the law of entropy as an essential supplement. At the same time, much progress has recently been made in the endeavor to base the law of entropy on the general principles of mechanics. For these reasons *a comprehensive discussion of all problems connected with the law of entropy* seems to be particularly desirable at the present time.

Such a discussion should embrace the following:

(1). The development of the empirical demonstrations of the law of entropy, together with a thorough digest of Carnot's works.

(2). An historical and critical discussion of the investigations bearing on the connection of the law of entropy with the general principles of mechanics.

(3). A complete report on all the applications of the law of entropy to the theory of physical and chemical processes.

The theses may be written in the German, Latin, French, or English language. They are to be sent to the Philosophical Faculty at Göttingen, together with a sealed letter which shall contain the name, profession and residence of the author, and be designated by the same motto as the thesis. The theses must be presented before August 31, 1890.

The award of the prizes will take place on March 11, 1891, the birthday of the founder, in a public session of the Philosophical Faculty.

The first prize amounts to 1700 M., the second to 680 M.

The prize-essays will remain the unrestricted property of their authors.

The title-page should contain an address to which the thesis may be returned in case it does not receive a prize.

The subjects for the prizes for the years 1888 and 1889 are announced in the Nachrichten der Königlichen Gesellschaft der Wissenschaften of Göttingen, 1886, No. 8, and 1887, No. 5.

GÖTTINGEN, April 1, 1888.

Sur les lignes géodésiques des surfaces à courbure constante.

Par R. Liouville.

L'objet de ce mémoire est d'indiquer la signification géométrique des équations différentielles du second ordre ayant leur intégrale générale linéaire par rapport aux constantes arbitraires et de former leurs invariants pour toutes les substitutions qui ne changent point, soit l'inconnue, soit la variable indépendante.

I. Les lignes géodésiques d'une surface à courbure constante peuvent être transformées toutes ensemble en un système plan de lignes droites, car, après application sur une sphère, les plans qui passent par le centre de cette dernière tracent sur celui qu'on a choisi toutes les lignes droites qui y sont contenues.

J'ajoute que chaque équation différentielle du second ordre, capable de représenter les droites d'un plan, c'est à dire ayant son intégrale générale linéaire par rapport aux constantes arbitraires, peut être regardée comme définissant les lignes géodésiques d'une surface à courbure constante. Soit en effet donnée ainsi

$$\alpha_1 z_1 + \alpha_2 z_2 + \alpha_3 z_3 = 0, \tag{1}$$

l'intégrale dont il s'agit; α_1, α_2, α_3 désignent des constantes arbitraires, z_1, z_2, z_3, des fonctions connues des variables x et y, entre lesquelles a lieu l'équation différentielle: le premier membre de (1) est la solution générale d'un système d'équations linéaires aux dérivées partielles du second ordre,

$$\left.\begin{aligned}
\frac{\partial^2 z}{\partial y^2} + P\frac{\partial z}{\partial x} + Q\frac{\partial z}{\partial y} + Tz = 0,\\
\frac{\partial^2 z}{\partial x\partial y} + P'\frac{\partial z}{\partial x} + Q'\frac{\partial z}{\partial y} + T'z = 0,\\
\frac{\partial^2 z}{\partial x^2} + P''\frac{\partial z}{\partial x} + Q''\frac{\partial z}{\partial y} + T''z = 0,
\end{aligned}\right\} \tag{2}$$

et l'on peut, dans celui-ci, multiplier l'inconnue z, par un facteur μ, sans que la relation (1) soit essentiellement modifiée. Si donc on fait en sorte que le produit

$$\mu^2 (z_1^2 + z_2^2 + z_3^2); \tag{3}$$

soit une constante c^2, les expressions $z_1\mu$, $z_2\mu$, ..., étant celles qui déterminent les points d'une sphère en coordonnées cartésiennes, et l'équation établie entre x et y n'ayant pas changé, il est mis en évidence qu'elle appartient aux lignes géodésiques d'une surface à courbure constante.

Ainsi la recherche de ces lignes équivaut précisément à l'étude d'une équation différentielle, dont l'intégrale ne contient qu'au premier degré les constantes arbitraires; comme on le montrerait sans peine, tout revient en ce cas à l'intégration d'une équation différentielle linéaire, du troisième ordre.

Lorsqu'on donne l'élément linéaire,

$$ds = \sqrt{edy^2 + 2fdxdy + gdx^2},$$

d'une surface, l'équation des lignes géodésiques s'en déduit par des formules déterminées et connues, mais la question inverse, notamment pour les surfaces à courbure constante, mérite quelques explications. Soit donc

$$y'' + a_1 y'^3 + 3a_2 y'^2 + 3a_3 y' + a_4 = 0, \tag{4}$$

l'équation des lignes géodésiques pour une surface de cette espèce; $a_1, a_2, \ldots, a_4$ sont fonctions d'x et d'y et vérifient des relations faciles à former. Pour trouver les expressions correspondantes de e, f, g, il convient d'obtenir d'abord le système, analogue à (2), corrélatif de l'équation (4) et dont trois intégrales satisfont à la condition

$$\Sigma z^2 = z_1^2 + z_2^2 + z_3^2 = 1. \tag{5}$$

Je supposerai que ce soit le système (2) lui-même et il en résultera évidemment

$$\Sigma\left(\frac{\partial z}{\partial x}\right)^2 = T'', \quad \Sigma \frac{\partial z}{\partial x}\frac{\partial z}{\partial y} = T', \quad \Sigma\left(\frac{\partial z}{\partial y}\right)^2 = T,$$

c'est à dire

$$ds^2 = T''dx^2 + 2T'dxdy + Tdy^2; \tag{6}$$

comme on connait l'un des systèmes qui peuvent être pris pour corrélatifs de l'équation (4), par exemple le suivant

$$\left.\begin{aligned}
&\frac{\partial^2 z}{\partial y^2} + a_1 \frac{\partial z}{\partial x} - a_2 \frac{\partial z}{\partial y} + \left(\frac{\partial a_2}{\partial y} - \frac{\partial a_1}{\partial x} + 2a_1a_3 - 2a_2^2\right) z = 0 = A(z),\\
&\frac{\partial^2 z}{\partial x\partial y} + a_2 \frac{\partial z}{\partial x} - a_3 \frac{\partial z}{\partial y} + \left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial x} + a_1a_4 - a_2a_3\right) z = 0 = A'(z),\\
&\frac{\partial^2 z}{\partial x^2} + a_3 \frac{\partial z}{\partial x} - a_4 \frac{\partial z}{\partial y} + \left(\frac{\partial a_4}{\partial y} - \frac{\partial a_3}{\partial x} + 2a_2a_4 - 2a_3^2\right) z = 0 = A''(z),
\end{aligned}\right\} \tag{7}$$

le point est de trouver le facteur μ, par lequel il faut multiplier ses intégrales pour justifier l'équation (5), ou, ce qui est la même chose, de déterminer, pour le système entièrement donné (7), l'expression de la somme

$$z_1^2+z_2^2+z_3^2=0;$$

or un calcul très simple fournit les quatre équations, aux dérivées partiellés du troisième ordre satisfaites par la fonction θ:

$$\begin{aligned}&\frac{\partial A''(\theta)}{\partial x}+2a_3.A''(\theta)-2a_4A'(\theta)+3\left(\frac{\partial a_4}{\partial y}-\frac{\partial a_3}{\partial x}+2a_2a_4-2a_3^2\right)\frac{\partial\theta}{\partial x}\\&\qquad+\theta\Big[\frac{\partial}{\partial x}\left(\frac{\partial a_4}{\partial y}-\frac{\partial a_3}{\partial x}+2a_2a_4-2a_3^2\right)+2a_3\left(\frac{\partial a_4}{\partial y}-\frac{\partial a_3}{\partial x}+2a_2a_4-2a_3^2\right)\\&\qquad-2a_4\left(\frac{\partial a_3}{\partial y}-\frac{\partial a_2}{\partial x}+a_1a_4-a_2a_3\right)\Big]=0,\\&\frac{\partial A'(\theta)}{\partial x}+a_2.A''(\theta)-a_4.A(\theta)+2\left(\frac{\partial a_3}{\partial y}-\frac{\partial a_2}{\partial x}+a_1a_4-a_2a_3\right)\frac{\partial\theta}{\partial x}\\&\qquad+\left(\frac{\partial a_4}{\partial y}-\frac{\partial a_3}{\partial x}+2a_2a_4-2a_3^2\right)\frac{\partial\theta}{\partial y}+\theta\Big[\frac{\partial}{\partial x}\left(\frac{\partial a_3}{\partial y}-\frac{\partial a_2}{\partial x}+a_1a_4-a_2a_3\right)\\&\qquad+a_2\left(\frac{\partial a_4}{\partial y}-\frac{\partial a_3}{\partial x}+2a_2a_4-2a_3^2\right)-a_4\left(\frac{\partial a_2}{\partial y}-\frac{\partial a_1}{\partial x}+2a_1a_3-2a_2^2\right)\Big]=0,\\&\qquad\text{etc.,}\end{aligned}$$

elles sont linéaires et admettent six intégrales communes; de l'une d'elles, on conclut un système d'expressions pour T, T', T'' ou e, f, g.

Ainsi, quand l'équation (4) est donnée, les coefficients e, f, g du carré de l'élément linéaire ne sont pas complètement déterminés; leurs expressions renferment six constantes arbitraires, qui y entrent d'une manière homogène. Soient (e_1, f_1, g_1), (e_2, f_2, g_2), deux systèmes distincts d'expressions pour les coefficients e, f, g et soient (x_1, y_1), (x_2, y_2), respectivement, les variables correspondantes: la substitution $(x_1, y_1; x_2, y_2)$ est visiblement l'une de celles, en nombre infini, qui transforment l'équation (4) en elle-même.

II. D'après ce qui précède, il est clair que les équations semblables à (4) sont toutes réductibles à celle-ci

$$y''=0$$

et par conséquent les unes aux autres; elles ne peuvent donc avoir d'invariants pour l'ensemble des transformations générales

$$x_1=\phi(x, y),\ y_1=\psi(x, y);$$

mais il n'en est pas de même si l'on se borne à changer, soit la variable, soit l'inconnue et, pour fixer les idées, c'est de ce dernier changement qu'il sera question, l'autre étant d'ailleurs exactement pareil.

Il convient d'appeler *invariants relatifs*, à l'égard des transformations

$$x_1 = x, \quad y_1 = \psi(x, y), \tag{8}$$

dont il s'agit, les fonctions des coefficients $a_1, a_2, \ldots, a_4$ et de leurs dérivées, qui se divisent par une puissance de

$$\frac{\partial\psi}{\partial y}$$

après la substitution (8); l'exposant de cette puissance est le poids de l'invariant considéré; s'il est nul, l'invariant est dit *absolu.* D'après ces deux formules,

$$y_1' = y' \frac{\partial\psi}{\partial y} + \frac{\partial\psi}{\partial x},$$

$$y_1'' = y'' \frac{\partial\psi}{\partial y} + y'^2 \frac{\partial^2\psi}{\partial y^2} + 2y' \frac{\partial^2\psi}{\partial x \partial y} + \frac{\partial^2\psi}{\partial x^2},$$

le coefficient de $y_1'^3$, dans l'équation transformée de (4), est

$$a_1 \left(\frac{\partial\psi}{\partial y}\right)^{-2};$$

a_1 est donc un premier invariant relatif, de poids égal à 2.

Une forme invariante ou canonique de l'équation proposée en résulte sans difficulté, car la fonction ψ définie par cette relation

$$\frac{1}{a_1}\left(\frac{\partial\psi}{\partial y}\right)^2 = \text{constante},$$

ou, avec plus de généralité, par la suivante

$$\frac{\partial\psi}{\partial y} = a_1^{\frac{1}{2}} f(x),$$

est manifestement un invariant absolu; lorsqu'on la prend pour inconnue dans l'équation (4), la transformée est canonique; ses coefficients jouent le rôle d'invariants absolus, mais ils dépendent d'une quadrature et ne sont point en conséquence des invariants *proprement dits*. Or certaines questions exigent que l'on distingue les invariants de cette espèce; il convient donc de chercher un système de fonctions, formées uniquement par combinaisons algébriques des coefficients et de leurs dérivées, données en outre de la propriété d'invariance.

C'est à quoi l'on peut parvenir en général comme je devais expliquer. z, $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$, sont des fonctions d'$x$ et d'y, déterminées par les relations (2); elles deviennent des fonctions d'une seule variable satisfaisant aux équations différentielles

$$dz - \left(\frac{\partial z}{\partial x} + y'\frac{\partial z}{\partial y}\right)dx = 0,$$

$$d\left(\frac{\partial z}{\partial x}\right) + \left\{\left(P''\frac{\partial z}{\partial x} + Q''\frac{\partial z}{\partial y} + T''z\right) + y'\left(P'\frac{\partial z}{\partial x} + Q'\frac{\partial z}{\partial y} + T'z\right)\right\}dx = 0, \quad (9)$$

$$d\left(\frac{\partial z}{\partial y}\right) + \left\{\left(P'\frac{\partial z}{\partial x} + Q'\frac{\partial z}{\partial y} + T'z\right) + y'\left(P\frac{\partial z}{\partial x} + Q\frac{\partial z}{\partial y} + Tz\right)\right\}dx = 0,$$

dès qu'on y remplace y par l'une des expressions qui vérifient l'équation (4). Je considère en particulier la combinaison

$$\frac{\partial z}{\partial y} + \lambda z = \zeta, \qquad (10)$$

où j'ai désigné par λ une fonction arbitraire d'x et d'y. Aucune des transformations (8) ne peut introduire $\frac{\partial z}{\partial x}$ dans le premier membre de (10) et il résulte du système (9),

$$d\zeta = \frac{\partial z}{\partial x}[\lambda dx - (P'dx + Pdy)] + (\zeta - \lambda z)[\lambda dy - (Q'dx + Qdy)]$$
$$+ z[d\lambda - (T'dx + Tdy)]. \qquad (11)$$

Par un choix convenable de λ, je puis rendre nul le coefficient de $\frac{\partial z}{\partial x}$, dans cette relation, ce qui exige

$$\lambda = P' + Py'; \qquad (12)$$

cela fait, les transformations (8) laisseront toutes à l'équation (11) le caractère d'une relation entre z et ζ seulement; ζ est donc, en vertu de (12), l'expression d'un invariant; il est visible de plus que z est un invariant absolu, $\frac{\partial z}{\partial y}$ un invariant relatif, de poids 1; il en est donc de même de λ. Ainsi, le premier membre de l'équation (12) et par suite la combinaison

$$P' + Py'$$

jouent le rôle d'invariant relatif, de poids égal à 1. Une autre combinaison de même espèce est connue avant toute recherche, c'est le déterminant fonctionnel

$$\Delta = \sum \pm z_1 \frac{\partial z_2}{\partial x}\frac{\partial z_3}{\partial y}$$

des trois solutions du système (2) et l'on sait que d'ailleurs on a

$$d \log \Delta = -[(P' + Q)\,dy + (P'' + Q')\,dx],$$

outre les relations, déjà signalées dans un Mémoire antérieur (Journal de l'Ecole Polytechnique, LVII° Cahier),

$$P = a_1, \quad 2P' - Q = 3a_2, \quad P'' - 2Q' = 3a_3, \quad Q'' = -a_4.$$

On en déduit d'abord

$$P' = a_2 - \frac{1}{3}\frac{\partial \log \Delta}{\partial y};$$

mais Δ peut être pris à volonté parmi les invariants relatifs de poids 1, sans que la correspondance établie entre l'équation donnée (4) et le système (2) soit troublée.

J'en profite pour poser

$$\Delta^2 = a_1;$$

l'expression (12) se change alors en la suivante

$$a_2 - \frac{1}{6}\frac{\partial \log a_1}{\partial y} + a_1 y' \tag{13}$$

et, si l'on divise par $a_1^{\frac{1}{2}}$ ce covariant de l'équation (4), le quotient n'est modifié en aucune manière par les transformations indiquées; pour obtenir une condition invariante, il suffira d'exprimer que ce quotient, multiplié par dx,

$$a_1^{\frac{1}{2}}dy + \left(a_2 a_1^{-\frac{1}{2}} - \frac{1}{6}\,a_1^{-\frac{1}{2}}\frac{\partial \log a_1}{\partial y}\right)dx,$$

est une différentielle exacte. Je trouve ainsi la relation

$$6a_1^2\frac{\partial a_1}{\partial x} + 2a_1\frac{\partial}{\partial y}\left(\frac{\partial a_1}{\partial y} - 6a_1a_2\right) - 3\frac{\partial a_1}{\partial y}\left(\frac{\partial a_1}{\partial y} - 6a_1a_2\right) = 0, \tag{14}$$

dont le premier membre ne peut être autre chose qu'un invariant relatif, et la substitution simple,

$$cy_1 = y$$

en montre immédiatement le poids, pourvu que c soit une constante. La conclusion est que cette combinaison

$$\nu_6 = 6a_1^2\frac{\partial a_1}{\partial x} + 2a_1\frac{\partial}{\partial y}\left(\frac{\partial a_1}{\partial y} - 6a_1a_2\right) - 3\frac{\partial a_1}{\partial y}\left(\frac{\partial a_1}{\partial y} - 6a_1a_2\right), \tag{15}$$

des coefficients de l'équation (4) et de leurs dérivées, est un invariant de poids égal à 6.

J'ai remarqué, dans un autre travail, une équation du second ordre, covariante de (4), pour toutes les transformations qui n'en changent pas la variable; voici cette équation

$$y'' - a_1 y'^3 - 3\left(a_1 - \frac{1}{3}\frac{\partial \log a_1}{\partial y}\right) y'^2 - 3\left(a_3 - \frac{2}{3}\frac{\partial \log a_1}{\partial x}\right) y' - \frac{1}{a_1}\left(\frac{\partial a_3}{\partial y} - 2\frac{\partial a_2}{\partial x} + a_1 a_4\right) = 0, \qquad (16)$$

à laquelle on peut donner le nom d'*adjointe* de la proposée, car il y a entre elle et le système adjoint à (2) la même correspondance qu'entre l'équation et le système (2). Il est clair qu'un nouveau covariant des équations (4) et (16) est leur différence, qui se réduit au premier ordre; on aura donc une condition invariante à satisfaire, si la valeur d'y' qui fait évanouir l'expression (13) doit annuler aussi cette différence, ce qui implique

$$\left(\frac{\partial a_1}{\partial y} - 6a_1a_2\right)\left(\frac{\partial a_1}{\partial x} - 3a_1a_3\right) - 6a_1^2\left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial x} + a_1a_4\right) = 0. \qquad (17)$$

On en conclut que le premier membre ν_5 de la relation précédente est un invariant relatif, de poids 5.

Enfin, l'on en trouve un troisième, en cherchant sous quelle condition l'équation différentielle

$$a_1 y' + a_2 - \frac{1}{6}\frac{\partial \log a_1}{\partial y} = 0$$

et une intégrale première de la proposée. L'expression qui doit alors être nulle et qui se représente ainsi

$$\begin{aligned}\nu_9 = 36a_1^3 \frac{\partial}{\partial x}\left(\frac{\partial a_1}{\partial y} - 6a_1a_2\right) + 6a_1\left(\frac{\partial a_1}{\partial y} - 6a_1a_2\right).\frac{\partial}{\partial y}\left(\frac{\partial a_1}{\partial y} - 6a_1a_2\right) \\ + \left(\frac{\partial a_1}{\partial y} - 6a_1a_2\right)^3 + 18a_1a_2\left(\frac{\partial a_1}{\partial y} - 6a_1a_2\right)^2 \\ - 12\left(\frac{\partial a_1}{\partial y} - 6a_1a_2\right)\left(\frac{\partial a_1}{\partial y} + 6a_1^2\frac{\partial a_1}{\partial x} - 9a_3a_1^3\right) + 6^3.a_1^5a_4,\end{aligned} \qquad (18)$$

est un invariant, de poids 9.

Je suppose d'abord que l'un au moins des invariants absolus

$$\nu_6 a_1^{-3},\ \nu_5^2 a_1^{-5},\ \nu_9^2 a_1^{-9} \qquad (19)$$

contienne la variable y; en le prenant pour inconnue, on met l'équation (4) sous forme invariante, sans qu'aucune quadrature ait été nécessaire. Ses coefficients sont alors des invariants absolus proprement dits, entre lesquels l'élimination d'y établit trois relations caractérisques.

Je suppose au contraire les trois expressions (19) fonctions de x seulement; en ce cas, il faut recourir à la transformée canonique déjà signalée, dans laquelle le coefficient d'y'^3 ne dépend que de x; je l'écris de cette manière

$$y'' + \alpha_1 y'^3 + 3\alpha_2 y'^2 + 3\alpha_3 y' + \alpha_4 = 0 \tag{20}$$

et je remarque d'abord que la relation

$$\nu_6 \alpha_1^{-3} = \phi_1(x),$$

jointe à celles-ci

$$\frac{\partial \alpha_1}{\partial y} = 0, \quad \frac{\partial \alpha_1}{\partial x} = \alpha_1',$$

entraine

$$2\frac{\partial \alpha_2}{\partial y} = \alpha_1' - \frac{\alpha_1}{6}\phi_1(x).$$

Si donc on a déterminé α_1, comme cela est permis, par la formule

$$6\alpha_1' - \alpha_1\phi_1(x) = 0,$$

on voit que α_2 ne contient pas y.

Ayant de plus $\nu_9 \alpha_1^{-\frac{9}{2}} = \phi_2(x)$,
ou, ce qui est la même chose,

$$\alpha_1^{\frac{3}{2}}\phi_1(x) + 6^3.(\alpha_2\alpha_1' - \alpha_1\alpha_2' + 2\alpha_2^3 - 3\alpha_1\alpha_2\alpha_3 + \alpha_1^2\alpha_4) = 0, \tag{21}$$

on connait, entre α_3 et α_4, une relation linéaire où n'entre que la variable x. Par suite, l'équation $\nu_5 \alpha_1^{-\frac{5}{2}} = \phi(x)$

qui a lieu aussi suivant l'hypothèse et qui s'écrit encore de cette façon

$$6\alpha_1^2\left(\frac{\partial \alpha_3}{\partial y} + \alpha_1\alpha_4 - \alpha_2'\right) + 6\alpha_1\alpha_2(\alpha_1' - 3\alpha_1\alpha_3) + \alpha_1^{\frac{5}{2}}\phi(x) = 0, \tag{22}$$

définit pour α_3 une fonction linéaire de y.

Soient L_1, L_2 les expressions, signalées ailleurs (C. R. de l'Acad. des Sc. de Paris, 28 nov. 1887), qui s'évanouissent quand l'équation (20) a son intégrale générale linéaire par rapport aux constantes arbitraires; α_1, α_2 ne renfermant pas y, la combinaison

$$\alpha_1 L_1 - \alpha_2 L_2$$

peut être représentée sous la forme

$$\alpha_1 L_1 - \alpha_2 L_2 = \frac{\partial}{\partial y}\left[\alpha_1 \frac{\partial \alpha_4}{\partial y} - \alpha_2 \frac{\partial \alpha_3}{\partial y} - 2\alpha_1 \frac{\partial \alpha_3}{\partial x} + \alpha_1\alpha_2\alpha_4 - 3\alpha_3(\alpha_2^2 + \alpha_1\alpha_3)\right] - \frac{\partial}{\partial x}\left[\alpha_2\alpha_1' - \alpha_1\alpha_2' + 2\alpha_2^3 - 3\alpha_1\alpha_2\alpha_3 + \alpha_1^2\alpha_4\right] \qquad (23)$$

comme on le prouverait sans peine. Or l'ensemble des termes compris dans la dernière parenthèse est une fonction de x; d'après l'équation (21), le premier membre de l'équation (23) est nul; les termes qui y figurent dans la première parenthèse constituent donc une fonction linéaire de y et cela ne se peut évidemment, si l'on n'a pas $\frac{\partial \alpha_3}{\partial y} = 0$.

Cette équation ayant lieu, α_3 et par suite α_4, c'est à dire tous les coefficients de l'équation (20) sont fonctions de la seule variable x, ce qui donne un cas déjà étudié et ramené aux quadratures. C'est le seul dans lequel la méthode indiquée ne conduise point à une transformée canonique à l'aide des invariants proprement dits, quand α_1 n'est pas nul, et il est clair qu'il devait échapper en effet à toute considération fondée sur l'emploi unique de ces invariants.

Si α_1 est nul, l'intégration dépend d'une équation différentielle, linéaire et du troisième ordre, *ne contenant aucun paramètre:* la formation de cette dernière est le moyen le plus commode d'étudier l'équation proposée; mais, ayant déjà traîté ce sujet, je n'ai point à en parler ici.

Dans le cas général, j'ai montré comment l'intégration de l'équation (4) se ramène aux opérations suivantes:

1°. Chercher une solution du système linéaire

$$\left.\begin{aligned} &\frac{\partial V}{\partial y} - P'V - PZ &&= 0, \\ &\frac{\partial Z}{\partial y} - Q'V - QZ + U &&= 0, \\ &\frac{\partial U}{\partial y} - T'V - TZ &&= 0, \end{aligned}\right\} \qquad (24)$$

dans lequel la variable x est regardée comme une constante;

2°. Intégrer une équation différentielle, linéaire et du 3° ordre, *où n'entre aucun paramètre.*

Or, on peut établir que la fonction Z joue le rôle d'un invariant relatif, de poids 1, pour toutes les transformations (8) et il en résulte que, si l'on considère l'équation déduite du système (24), à laquelle satisferait l'inconnue

$$Za_1^{-\frac{1}{2}},$$

isolée des deux autres, ses invariants absolus, sont, au sens même qui a été donné à ce mot par Mr. Halphen, des invariants de l'équation proposée (4), tels que nous les avons définis.

Deux cas surtout sont dignes de remarque.

Le premier se présente quand toutes les relations entre ces invariants demeurent indépendantes d'y; les solutions du système (24) s'obtiennent alors par des quadratures;

Le second a lieu, quand toutes ces relations sont au contraire indépendantes de x; le système (24), dont il faut avoir une intégrale, peut se réduire alors à une équation différentielle, du troisième ordre et linéaire, *ne renfermant aucun paramètre,* de sorte que tout le problème aboutit à des équations de cette nature.

PARIS, le 20 avril 1888.

On the Primitive Groups of Transformations in Space of Four Dimensions.

By James M. Page.

In a series of papers published at various intervals since 1873 Sophus Lie has developed a new mathematical theory, which he calls the "Theory of Groups of Transformations." His researches are nearly related to several other branches of mathematics, especially to the "Theory of Substitutions" of Galois, to the Modern Geometry and Modern Algebra, and to the Theory of Differential Equations. Lie has very materially modified this last-named theory by basing it upon his Theory of Groups, and by showing that this is the natural and correct starting point for a Theory of Differential Equations.

But, in order to derive practical advantage from the Theory of Groups for the Theory of Differential Equations and its kindred branches of mathematics, it is necessary to know some, and if possible, *all* groups of transformations. Lie has already given methods for finding all of certain classes of groups in any number of variables, although to carry out the necessary calculations in some cases may be practically impossible. However, he has actually calculated all groups in one, two, and three variables. It will be the object of this paper to find all of a certain class of groups—the so-called *primitive* ones—in *four* variables.

As Lie's theory, since he has not yet published a connected work on the subject, is not generally known, we shall collect the principal definitions and theorems of the same needed in this paper, in an

Introduction.

A. We shall always operate with "infinitesimal point transformations," inasmuch as they are most convenient for our calculations, and Lie has shown how to find the groups of finite transformations by very simple means, when we know the groups of infinitesimal transformations.

Lie defines an infinitesimal point transformation in n variables by the equations

$$x_i' = x_i + \xi_i(x_1 \ldots . x_n)\,\delta t, \qquad i = 1 \ldots . n, \tag{α}$$

where δt is an infinitesimal quantity, and where ξ_i are "analytical functions" of their arguments, in Weierstrass' sense of the word.

For the infinitesimal transformation (α), Lie has introduced the symbol

$$X(f) \equiv \sum_1^n {}^i\, \xi_i(x_1 \ldots . x_n)\frac{\partial f}{\partial x_i},$$

or, when we write as is customary,

$$p_i \equiv \frac{\partial f}{\partial x_i},$$

the symbol is

$$X(f) \equiv \sum_1^n {}^i\, \xi_i(x_1 \ldots . x_n)\,p_i.$$

From this symbol we can see at a glance the form of (α); for if f is any function of $x_1 \ldots . x_n$, we have

$$f(x_1' \ldots . x_n') = f(x_1 \ldots . x_n) + X(f)\,\delta t,$$

from which follows

$$x_i' = x_i + \xi_i(x_1 \ldots . x_n)\,\delta t.$$

It is easy to see that the symbol $X(f)$, or as we shall for convenience simply write it, Xf, is independent of the choice of variables.*

If

$$X_1f,\ X_2f \ldots . X_rf$$

are r infinitesimal transformations, they are said to be *independent*, when it is impossible to choose r such constants c_k as make the expression

$$c_1X_1f + c_2X_2f + \ldots . + c_rX_rf$$

vanish identically.

When we perform the operation

$$X_i(X_k(f)) - X_k(X_i(f)) \equiv \sum_1^n {}^j \{X_i(\xi_{jk}) - X_k(\xi_{ji})\}\frac{\partial f}{\partial x_j},$$

---

[*] If the infinitesimal transformation Xf is repeated an infinite number of times, we get ∞^1 finite transformations

$$x_i' = x_i + t.X(x_i) + \frac{t^2}{1.2}.XX(x_i) + \ldots .,$$

where t is an arbitrary parameter.

we say that we *combine* the two infinitesimal transformations X_if and X_kf. For this operation we shall write Jacobi's simple symbol

$$(X_iX_k).$$

Now Lie has given the fundamental

Definition: "If r independent infinitesimal transformations $X_1f \ldots . X_rf$ satisfy in pairs all relations of the form

$$(X_iX_\kappa) \equiv \sum_{1}^{r}{}_{s}\, c_{iks}X_s(f), \qquad i,\, k = 1 \ldots . r,$$

where the c_{iks} are constants, then form the infinitesimal transformations

$$\sum_{1}^{r}{}_{k}\, c_kX_kf,$$

where the c_k are constants, a group with r members."

The system of constants c_{iks} is called the *composition* (Zusammensetzung) of our group.

Any three transformations X_if, X_jf, X_kf must satisfy the celebrated identity of Jacobi,

$$(X_i(X_jX_k)) + (X_j(X_kX_i)) + (X_k(X_jX_i)) \equiv 0,$$

and this gives relations which the c_{iks} must satisfy, if they form the composition of a group, viz:

$$\sum_{1}^{r}{}_{s}\, (c_{ijs}\cdot c_{sk\nu} + c_{jks}\cdot c_{is\nu} + c_{kis}\cdot c_{sj\nu}) = 0,$$

$$(i,\, k,\, j,\, \nu = 1 \ldots . r).$$

Lie has shown, conversely, that if we have a system of c_{iks} which satisfy the equations last written, there are always groups with r members which possess this composition.

If all $$c_{iks} = 0,$$

we say that our above transformations are *in pairs commutable* (paarweise vertauschbar), or simply *commutable.* We may take any r independent infinitesimal transformations of our above group, say $X_1f \ldots . X_rf$, as representatives of that group, since all of its infinitesimal transformations are included in the general infinitesimal transformation of the group

$$\sum_{1}^{r}{}_{k}\, c_k'X_kf, \quad (c_k' \text{ const.}).$$

and when we know $X_1 f \dots X_r f$, we may at once write all other transformations of the group.

For a group of transformations of r members we shall sometimes use the symbol G_r.

If among all the infinitesimal transformations

$$\sum_1^r {}_k \, c_k X_k f$$

of our G_r, there are some, say

$$X_1 f \dots X_{r-\rho} f,$$

which form themselves a group, we call this group a *subgroup* (Untergruppe) of the original G_r. If the G_r contains a subgroup

$$Y_k f \equiv d_{k1} X_1 f + \dots + d_{kr} X_r f, \qquad k = 1 \dots r - \rho,$$

where relations hold of form

$$(Y_k X_i) \equiv \sum_1^{r-\rho} {}_s \, d_{iks} Y_s f, \qquad i, k = 1 \dots r,$$

then is the subgroup $Y_1 f \dots Y_{r-\rho} f$ said to be *invariant* in the G_r $X_1 f \dots X_r f$; and the $Y_k f$ are said to form an *invariant subgroup*.

The finding of all subgroups of a given G_r involves only the performance of algebraic operations.

Two groups are said to be *similar* (ähnlich) when by a proper choice of independent variables the one group can be transformed to the other; that is, if we have the two G_r,

$$X_1 f \dots X_r f \text{ and } X_1' f \dots X_r' f,$$

they are *similar*, when it is possible to introduce into $X_k f$ such new independent variables that relations hold of the forms

$$X_k f = \sum_1^r {}_i \, c_{ki} X_i' f, \qquad k = 1 \dots r,$$

where c_{ki} are constants.

We shall consider all groups *known*, which are similar to a given group.

B. For a space of n dimensions we shall use the customary symbol R_n; and for a manifoldness of k dimensions, M_k.

In all operations with infinitesimal transformations we shall be satisfied with a *first approximation* to accuracy; that is, we shall drop out of consideration infinitesimals of the second and higher orders.

Now a function $$\phi(x_1 \dots . x_n)$$

is transformed by the infinitesimal transformation Xf, in the same variables, to

$$\phi + X(\phi)\,\delta t.$$

We say that the function ϕ *admits of* (gestattet) the infinitesimal transformation Xf, or is *invariant under* Xf, when

$$X(\phi) \equiv 0,$$

i. e. when ϕ is a solution of the linear partial differential equation of the first order, $$Xf = 0.$$

If in the ordinary XY-plane, a function $\phi(xy)$ admits of each infinitesimal transformation of a G_r, it is evident that each of the ∞^1 curves

$$\phi(xy) = \text{const.}$$

is unchanged by the transformations of the group; that is, each curve of the family is *invariant* under the group. Thus each ordinary point of the plane must be simply moved up on one of the curves by the transformations of the group. In this case the G_r is said to be *intransitive.* If, on the contrary, a G_r in xy can transform every ordinary point of the plane to every other ordinary point of the plane, the G_r is said to be *transitive.*

Similarly in R_n, if a function $\phi(x_1 \dots . x_n)$ admits of every infinitesimal transformation of a group, it is evident that each of the ∞^1 M_{n-1},

$$\phi(x_1 \dots . x_n) = \text{const.}$$

is unchanged by the transformations of the G_r, is *invariant* under the G_r. Here every point of general position in R_n is moved up on one of these M_{n-1} by the transformations of our G_r. In this case our group in n variables is said to be *intransitive.*

If, on the contrary, every ordinary point in R_n can be carried by the transformations of our G_r to every other such point in R_n, we call the group *transitive.* Analytically expressed, the group

$$X_1 f \dots . X_r f$$

is *transitive*, when the equations

$$X_1 f = 0 \dots . X_r f = 0$$

have no common solution.

But each member of the family of curves

$$\phi\,(xy) = \text{const.}$$

in the plane, or, as we may write it,

$$\Psi\,(\phi) = \text{const.},$$

where Ψ is an arbitrary function of ϕ, need not be invariant for the whole family to be invariant. The family as a whole is still invariant when its various members are swapped around among each other by the transformations of our G_r. Now we know

$$\phi\,(xy) = \text{const.}$$

goes by means of the infinitesimal transformation Xf to

$$\phi + X(\phi)\,\delta t = \text{const.}$$

Thus if

$$X(\phi) \equiv \Psi\,(\phi)$$

the family of curves is as a whole evidently invariant.

This family of curves is defined by a linear partial differential equation IO,

$$Af \equiv X\frac{\partial f}{\partial x} + Y\frac{\partial f}{\partial y} = 0,$$

and when the family is invariant under Xf, the differential equation is also said to be invariant under that transformation.

The analytical criterium that

$$Af = 0$$

should be invariant under Xf is

$$(X, A) \equiv \rho\,(xy).Af,$$

where ρ is some function of x and y.

Now a group $X_1 f \ldots . X_r f$

in the plane is said to be *imprimitive* when it leaves a family of ∞^1 curves,

$$\phi\,(xy) = c$$

invariant, *i. e.* when

$$X_i\,(\phi) = \Psi_i(\phi), \qquad i = 1 \ldots . r.$$

Otherwise is the group said to be *primitive.*

Similarly, in R_n the $\infty^{n-q} M_q$,

$$\phi_1(x_1 \ldots . x_n) = c_1, \ldots . \phi_{n-q}(x_1 \ldots . x_n) = c_{n-q}$$

are said to be invariant as a whole under the transformation Xf, in the same variables, when

$$X(\phi_i) = \Omega_i\,(\phi_1 \ldots . \phi_{n-q}),$$

where the Ω are some functions of $\phi_1 \ldots \phi_{n-q}$. These M_q are represented in R_n by a system of q linear partial differential equations IO,

$$Y_1 f = 0, \ldots Y_q f = 0,$$

where

$$Y_k f \equiv \sum_1^n {}_i \chi_{ki}(x_1 \ldots x_n) \frac{\partial f}{\partial x_i};$$

and for these $Y_k f$ must hold the relations

$$(Y_i Y_j) = \sum_1^q {}_s \psi_{ijs}(x_1 \ldots x_n) Y_s f.$$

Such a system is called a "complete (vollständig)* system with q members."

It may be shown that a complete system with q members in n variables always has $(n-q)$ independent solutions.

The analytical criterium that this system should be invariant under the transformation Xf in the same variables is,

$$(X, Y_i) = \sum_1^q {}_s \rho_{is}(x_1 \ldots x_n) Y_s f,$$

where the ρ are some functions of $x_1 \ldots x_n$.

Now we say that a group $X_1 f \ldots X_r f$ in n variables is *imprimitive*, when it leaves a family of $\infty^{n-q} M_q$ (which fill R_n exactly once),

$$\phi_1(x_1 \ldots x_n) = c_1 \ldots \phi_{n-q}(x_1 \ldots x_n) = c_{n-q}$$

invariant, *i. e.* when

$$X_i(\phi_k) = \Omega_{ki}(\phi_1 \ldots \phi_{n-q}), \quad \begin{cases} k = 1 \ldots n-q \\ i = 1 \ldots r \end{cases},$$

where the Ω are some functions of $\phi_1 \ldots \phi_{n-q}$. Otherwise is the group said to be *primitive.*

If we hold any ordinary point in the plane, the directions through it are transformed among each other by the transformations of any group. If now the group leaves a family of ∞^1 curves,

$$\phi(xy) = c$$

invariant, one such curve goes through each ordinary point in the plane; and when we hold such a point, the curve, and its tangential direction through that point, must remain invariant with the point. Lie has shown, *vice versa*, that a group in the plane is *imprimitive*, when with each ordinary point which we hold, an invariant direction through the point is connected—a necessary and sufficient criterium.

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* The *Theory of the Complete System* was developed by Jacobi and Clebsch.

Thus, a group in the plane is *primitive* when with each ordinary point which we hold, no invariant direction is connected.

Similarly in R_n, if we have a group which leaves a family of $\infty^{n-q}M_q$ invariant, through each ordinary point goes a pencil of ∞^{q-1} directions, which are transformed among each other by the transformations of the group, when we hold the point.

Lie has shown that if with each ordinary point in R_n such a pencil of ∞^{q-1} directions is invariantly connected, a system of q linear partial differential equations IO is invariant. If this system is *complete*, then it is evident that the group in question is imprimitive; if the system is not complete, the group *may* be primitive.

Applying the above to $n=4$, we see that in R_4 there are *three* cases: A single direction may be invariant with each ordinary point; or a pencil of ∞^2 directions, and no single direction, may be invariant; or a pencil of ∞^1 directions—and no single direction, and no pencil of ∞^1 directions—may be invariant. In the first case the groups are evidently imprimitive; at any rate, it is easy to see that they must be so. Lie has shown that they are also in the second case imprimitive. In the third case, however, the groups *may* be primitive.

C. From now on we shall confine ourselves to four variables.

In the infinitesimal transformation

$$Xf \equiv \sum_{1}^{4}{}^{i}\, \xi_i(x_1 \dots . x_4)\, p_i$$

the ξ_i are analytical functions. Thus, if x_i^0 is an ordinary point in R_4, the ξ_i may be expanded in its neighborhood in powers of $x_i - x_i^0$. Let us choose the origin as an ordinary point, and then we may expand the ξ_i in its neighborhood in powers of x_i. Thus we may write for the neighborhood of the origin

$$Xf \equiv \sum_{1}^{4}{}^{i}\, \alpha_i p_i + \sum_{ik}^{1\dots4} \beta_{ik} x_i p_k + \sum_{ikj}^{1\dots4} \gamma_{ikj} x_i x_k p_j + \dots .$$

Here α, β .... are constants, some of which may be zero. If the constant coefficients of all terms up to the $(k-1)^{\text{st}}$ vanish, that is, if the transformation Xf begin with terms of the k^{th} degree, we say *Xf is of the k^{th} order.* We call the terms of the k^{th} degree of a transformation of the k^{th} order the *initial terms* of the transformation.

It is easy to see that a transformation of the i^{th} order combined with a transformation of the k^{th} order gives a transformation of at least the $(i+k-1)$ order.

We may remark that all transformations of the forms $\sum_{1}^{4}{}_{i}\, a_i p_i$ are *mere translations*, and $\sum_{ik}^{1\ldots 4} b_{ik} x_i p_k$ *linear homogeneous transformations.*

Our object is to find all groups in R_4 which transform the directions through an ordinary point, which we hold, so that no direction and no plane pencil of ∞^2 directions is invariant. Having chosen the origin as an ordinary point, let us hold it, and see how the directions through it are transformed by a transformation Xf. If we expand Xf in the neighborhood of the origin, we see that the most general transformation which leaves this point invariant has the form

$$\bar{X}f \equiv \sum_{ik}^{1\ldots 4} \beta_{ik} x_i p_k + \sum_{ikj}^{1\ldots 4} \gamma_{\,kj} x_i x_k p_j + \ldots .$$

But $\bar{X}f$ does transform the points, very near the origin, with the coordinates dx_i. Any such point dx_i^0, together with the origin, determines a direction through the origin; and these directions are transformed just as the points very near the origin dx_i are. But we see, when we drop infinitesimals of the second and higher orders, that $\bar{X}f$ transforms the points dx_i, that is, the directions through the origin, just as

$$Yf \equiv \sum_{ik}^{1\ldots 4} \beta_{ik} x_i p_k$$

does; that is, the directions through the origin are transformed by a linear homogeneous transformation. If now we have a G_r,

$$X_1 f \ldots . X_r f,$$

Lie has proved that all transformations of the G_r which leave a point invariant, form a subgroup of the original G_r. We have seen that all primitive groups are also transitive. Thus in the G_r which we seek there must always be four infinitesimal transformations of the zero order:

$$p_k + \ldots . , \qquad k = 1 \ldots . 4,$$

otherwise would every ordinary point like the origin be translated on a manifoldness by the transformations of our group; that is, the group would be intransitive. Thus in a primitive G_r in R_4 there must always be $(r-4)$ independent infinitesimal transformations of the form

$$\overline{X}_\rho f \equiv \sum_{ik}^{1\dots 4} \beta_{ik\rho} x_i p_k + \sum_{ikj}^{1\dots 4} \gamma_{ikj\rho} x_i x_k p_j + \dots\dots,$$

$$\rho = 1 \dots\dots r-4,$$

and the $\overline{X}_\rho f$ must satisfy relations of the form

$$(\overline{X}_i \overline{X}_k) = \sum_{1}^{r-4} {}_s\, c_{iks} \overline{X}_s f. \qquad (\gamma)$$

The directions through the origin will be transformed by

$$Y_\rho f \equiv \sum_{ik}^{1\dots 4} \beta_{ik\rho} x_i p_k, \qquad \rho = 1 \dots\dots r-4,$$

exactly as by $\overline{X} f_1 \dots\dots \overline{X}_{r-4} f$; and if we substitute for $X_1 f \dots\dots X_{r-4} f$ their values in (γ), we find

$$(Y_i Y_k) \equiv \sum_{1}^{r-4} {}_s\, d_{iks} Y_s f, \qquad i, k = 1 \dots\dots r-4.$$

That is, our primitive G_r in R_4 transforms the directions through the origin when we hold this point just as the initial terms of its transformations IO do. These initial terms of the transformations IO form, taken by themselves, a group of linear homogeneous transformations.

Thus to attack the problem of finding all primitive G_r in R_4 we shall first find all linear homogeneous G_r, $Y_k f$, in R_4, which leave no direction and no pencil of ∞^2 directions through an ordinary point, which we hold, invariant. Then we take the transformations of these linear homogeneous groups as initial terms of the transformations *IO* of the primitive G_r which we seek.

But since the $Y_k f$ are linear and homogeneous in $x_1 \dots\dots x_4$, we may consider the $x_1 \dots\dots x_4$ as homogeneous coordinates in R_3. Thus to find the *linear homogeneous* G_r in R_4, we must write the *projective* groups in R_3 homogeneous and linear in $x_1 \dots\dots x_4$.

We shall then choose those of these linear homogeneous G_r which leave no direction, and no pencil of ∞^2 directions, through an ordinary point which we hold, invariant, and determine the combinations of the primitive G_r and the G_r themselves.

In calculating our primitive G_r we shall make use of an important theorem of Lie. He has proved that *if we can give a* $\overline{G}_r$ *which has the same combination as a required* G_r, *and if the initial terms of the transformations of the two groups are the same, then are the two groups* SIMILAR.

Suppose now we have a G_r in R_4,

$$X_1f \dots X_rf,$$

where the transformations have been developed in series, so that

$$X_kf \equiv \sum_1^4 {}_j\, \xi_{kj}^{(\lambda)} p_j + \dots,$$

where we only write the initial terms on the right hand. Then the expressions

$$\overline{X}_kf \equiv \sum_1^1 {}_j\, \xi_{kj}^{(\lambda)} p_j$$

form themselves a G_r. If now

$$(\overline{X}_i\overline{X}_k) \equiv \sum_1^r {}_s\, c_{iks}\overline{X}_sf,$$

and at the same time

$$(X_iX_k) = \sum_1^r {}_s\, c_{iks}X_sf$$

where c_{iks} are *the same* constants in both equations, *then we say that* $X_1f \dots X_rf$ *are connected by normal relations.*

D. Lie has found all projective G_r in R_3 and has classified them according to the figures they leave invariant. He distinguishes

(1). The general projective G_r which leaves no figure invariant.
(2). " " " " " plane "
(3). " " " " " linear complex invariant.
(4). " " " " " straight line "
(5).* " " " " " F_2 "
(6). " " " " " twisted curve *IIIO* invariant.
(7). " " " " " point invariant.

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[*] "F_2" is sometimes used for "Surface of second degree."

Lie has already found the primitive G_r in R_4 given by cases (1) and (5); we give them at the end of this paper.

We may exclude case (7) at once, since that evidently assigns to each point of general position, which we hold, an invariant direction, so that the groups are imprimitive. Further, it is easy to prove by the theory of Pfaff's Problem, as Lie has shown, that case (2) gives only imprimitive G_r.

Thus we have only the cases (3), (4), and (6) to discuss. The cases (3) and (6) give only one projective G_r each, which assign to a point of general position, which we hold, no invariant direction. But there are a large number of subgroups of the general G_r in case (4) which may give primitive G_r in R_4.

Thus our first object will be to find these subgroups in case (4).

E. The general projective group in R_3 which leaves a straight line which we may choose as $z = \text{const.} = c$, invariant,

has the form

$$\begin{aligned} &p,\ q,\ zp,\ zq \ldots\ldots (T_k), \\ &xq,\ xp - yq,\ yp \ldots\ldots (U_k), \\ r,\ xp + yq + 2zr,\ &xzp + yzq + z^2 r \ldots\ldots (V_k), \\ &xp + yq \ldots\ldots (W). \end{aligned}$$

We see that the transformations T_k are in pairs commutable, and that they leave each of the planes $z = c$, and each point on the straight line $z = c$, absolutely invariant. Further, the U_k form a group which leaves the planes $z = c$ singly invariant, but transforms the points on the straight line $z = c$. We see that the T_k form an invariant subgroup of the group U_k, T_k.

The V_k form a group, of the same combination as the U_k, which leaves the points on the straight line $z = c$ invariant, but transforms the planes $z = c$. And the T_k also form an invariant subgroup of T_k, V_k. U_k and V_k, by the Principle of Duality, are equally privileged groups.

The transformation W leaves the points on the line $z = c$ as well as the planes $z = c$ both absolutely invariant. We see that T_k, U_k, V_k form a G_{10}—a subgroup of our G_{11}—of which each transformation is absolutely invariant under W.

From what we said above we see that the subgroups which we seek must leave no point or plane in R_3 invariant. Thus:

We wish to find all subgroups of the above G_{11} *which leave the straight line* $z = c$, *but no point or plane, invariant.*

Consider two transformations of our G_{11}:

$$\sum^{i}(\alpha_{ki}T_i+\beta_{ki}U_i+\gamma_{ki}V_i+\delta_k W),$$
$$\sum^{j}(\alpha_{ji}T_i+\beta_{ji}U_i+\gamma_{ji}U_i+\delta_j W).$$

If we combine these two we get a transformation in which the new β depend only on the old β, and the new γ on the old γ. No δ occurs in the new transformation. Thus we conclude that

$$\sum^{i}(\beta_{ki}U_i+\gamma_{ki}V_i+\delta_k W)$$

form themselves a group.

Further, we see that

$$\sum^{i}\beta_{ki}U_i,\ \sum^{i}\gamma_{ki}V_i$$

form also a group

If we compare the group $\sum^{i}\beta_{ki}U_i$ with the original G_{11}, we see that the two groups transform the points on the line $z=c$ in exactly the same manner; and also the group $\sum^{i}\gamma_{ki}V_i$ transforms the planes $z=c$ just as the G_{11} does.

Thus if the group $\sum^{i}\beta_{ki}U_i$ contains less than three members, a point on the straight line $z=c$ will be invariant,* which case is excluded. Same way must the group $\sum^{i}\gamma_{ki}V_i$ contain three members, otherwise we would have an invariant plane.

Thus we wish to find all subgroups of the group

$$\sum^{i}\beta_{ki}U_i,\ \sum^{i}\gamma_{ki}V_i$$

which transform the above-mentioned points and planes so that no point or plane is invariant. Hence, since all the U must occur, we may write these subgroups,

$$U_1+\sum_{1}^{3}{}^{i}\alpha_{1i}V_i,\ U_2+\sum_{1}^{3}{}^{i}\alpha_{2i}V_i,\ U_3+\sum_{1}^{3}{}^{i}\alpha_{3i}V_i,\ \sum^{i}\beta_i V_i.$$

But since all the V must occur, we may write the required groups,

$$V_1+\sum_{1}^{3}{}^{i}\delta_{1i}U_i,\ V_2+\sum_{1}^{3}{}^{i}\delta_{2i}U_i,\ V_3+\sum_{1}^{3}{}^{i}\delta_{3i}U_i,\ \sum^{i}\gamma_i U_i.$$

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* By a theorem of Lie's on the groups of the M_1.

Since U and V are equally privileged, we easily see that the only possibilities here are the two groups

$$V_1 + U_1, \; V_2 - U_2, \; V_3 - U_3, \text{ and } V_1, \; V_2, \; V_3, \; U_1, \; U_2, \; U_3.$$

We shall call these two our "abridged" (verkürzte) groups. The G_r which we seek must all contain the transformations of one of these abridged groups. Since the T_k and W transform neither the points nor the planes mentioned above, these transformations may occur in the required G_r free, or added with constant coefficients to the transformations of one of the abridged groups.

(A). Let us take up the case where the transformations of the first of the above abridged groups occur in our sought subgroups; that is, the transformations of $V_1 + U_1, \; V_2 - U_2, \; V_3 - U_3$ occur.

(1). Suppose for the present no transformation W occurs at all; and also no free transformation of the form $\sum_{1}^{4}{}^{\mu} \delta_\mu T_\mu$. Then the transformations of our sought groups can only have the forms

$$V_1 + U_1 + \sum{}^{i} \alpha_i T_i, \; V_2 - U_2 + \sum{}^{j} \beta_j T_j, \; V_3 - U_3 + \sum{}^{k} \gamma_k T_k.$$

Let us introduce new variables into these transformations by means of the substitution

$$x' = x - \alpha_1 z + \alpha_2, \quad y' = y + \frac{\beta_2}{2}.$$

Since nothing depends upon the symbols we use, we can drop the accents from the new transformations, and our substitution is equivalent to putting

$$\alpha_1 = \alpha_2 = \beta_2 = 0.$$

We now easily find by combining our transformations, and remembering that they must form a group, that the only possible group is

$$r + xq, \; 2(yq + zr), \; xzp + yzq + z^2 r - yp.$$

It is easy to see that this projective group leaves the F_2,

$$y - zx = 0$$

and all generators of one family, absolutely invariant.

If the transformation W occurs, it is easy to see that it cannot occur additively, *i. e.* only free. Hence we get the group

$$r + xq, \; xp + yq, \; 2(yq + zr), \; xzp + yzq + z^2 r - yp.$$

This group leaves the F_2,

$$y - zx = 0$$

and two different generators of one family invariant.

(2). Suppose a transformation of the form

$$\sum_{1}^{4}{}^{\mu}\, \delta_\mu T_\mu$$

occurs free. Suppose for the present that no W occurs; our transformations are

$$r + xq + \sum{}^{i} \alpha_i T_i,\ \ 2(yq + zr) + \sum{}^{j} \beta_j T_j,\ \ (xz - y)p + yzq + z^2 r + \sum{}^{k} \gamma_k T_k$$

$$\sum \delta_\mu T_\mu \equiv \delta_1 T_1 + \delta_2 T_2 + \delta_3 T_3 + \delta_4 T_4.$$

By making the same change of variables as in (1), we can again put

$$\alpha_1 = \alpha_2 = \beta_2 = 0.$$

By combination we find that the transformations

$$(\delta_2 + \delta_3)(p - zq) + (\delta_4 - \delta_1)(q + zp),$$
$$\delta_2 q - \delta_3 zp$$

must occur.

(I). Suppose $\delta_2 = -\delta_3$ and

(a). $$\delta_3 = 0.$$

Then we see
$$(\delta_4 - \delta_1)(q + zp)$$
must occur.

(i). Suppose $\delta_1 = \delta_4$. Then must $\delta_1 \times \delta_4 \neq 0$. Hence

$$p + zq$$

occurs. Any other transformation $\sum{}^{i} a_i T_i$, which occurs has the form

$$ap + bq + czp.$$

Combine this transformation with $r + xq + \sum{}^{i} \alpha_i T_i$, and we see

$$c\,(p - zq) - aq,$$

and
$$2cq$$
occur. If here $c \neq 0$ we see

$$p,\ q,\ zp,\ zq,$$

must all occur. Thus we find the group

$$r + xq,\ yq + zr,\ (xz - y)p + yzq + z^2r,\ p,\ zp,\ q,\ zq.$$

If the transformation W occurs, we see that it can only occur free, and we find a group of which the last is a subgroup,

$$r + xq,\ yq + zr,\ (xz - y)p + yzq + z^2r,\ xp + yq,\ p,\ zp,\ q,\ zq.$$

It is easy to see that these groups leave two coinciding straight lines invariant.

If above we have $c = 0$ and $a \neq 0$, we find these groups again. Thus

$$c = a = 0,$$

and we see then

$$b = 0.$$

Our transformations now have the forms

$$r + xq + \alpha_3 zp + \alpha_4 zq,\ 2(yq + zr) + \beta_3 zp + \beta_4 zq,$$

$$(xz - y)p + yzq + z^2r + \sum_{2}^{4}{}^{i}\,\gamma_i T_i,\ p + zq.$$

By combination we easily find

$$\gamma_i = \alpha_3 = \beta_4 = 0,\ \beta_3 = -\alpha_4.$$

If now $\alpha_4 = 0$ we find the group

$$r + xq,\ yq + zr,\ (xz - y)p + yzq + z^2r,\ p + zq.$$

If $\alpha_4 \neq 0$, introduce new variables by means of the substitution

$$x' = \frac{x}{\alpha_4} + \frac{z}{2},\ y' = \frac{y}{\alpha_4},$$

and we find our last group again.

This group leaves the surface of second degree

$$y - zx = 0$$

and two coinciding generators of the same invariant.

If the transformation $xp + yp$ occur, it can only occur free, which gives the group

$$r + xq,\ yq + zr,\ (xz - y)p + yzq + z^2r,\ p + zq,\ xp + yq.$$

This group leaves a surface second degree invariant, and also one of its generators.

(ii). Suppose now $\delta_1 \neq \delta_4$.

Thus we see $q + zp$ occurs. Combine this transformation with $r + xq + \sum^i \alpha_i T_i$ and we see

$$p - zq, q$$

occur also.

We see at once that no other transformation $\sum^\mu \delta_\mu T_\mu$ can occur, unless *all* occur, a case which we have already finished.

We easily find the group

$$r + xq, yq + zr, (xz - y)p + yzq + z^2 r, q, zp, p - zq.$$

The transformation $xp + yq$ may occur free, which gives a group of which the last is a subgroup,

$$r + xq, yq + zr, (xz - y)p + yzq + z^2 r, xp + yq, q, zp, p - zq.$$

Lie has shown that these groups leave a linear complex and one of its straight lines invariant.

We now come to the case

(b). $\delta_2 = -\delta_3, \ \delta_2 . \delta_3 \neq 0.$

We see at once that the transformation $q + zp$ occurs. Combine this with $r + xq + \sum^i \alpha_i T_i$ and then we find that also

$$p - zq, q$$

must occur. Thus we have the same groups as in the last case.

(II). $\delta_2 \neq -\delta_3.$

Thus,

$$p - zq + k(q + zp), \quad k = \frac{\delta_4 - \delta_1}{\delta_2 + \delta_3}$$

occurs; also

$$\delta_2 q - \delta_3 zp.$$

(a). If now $\delta_3 \neq 0$, then $zp - \frac{\delta_2}{\delta_3} q$ occurs. Combine with

$$r + xq + \sum^i \alpha_i T_i$$

and we see

$$p - zq, q$$

occur. Thus we have again the same case as above.

(b). If $\delta_3 = 0$, then $\delta_2 \neq 0$ and q must occur, with $p - zq + k.zp$. If $k \neq 0$, we see that we get the above case again.

If $k = 0$, we have

$$q, p - zq.$$

Combine the last transformation with

$$(xz - y)p + yzq + z^2 r + \sum^i \gamma_i T_i$$

and we see that we get still our often-mentioned case above.

These are all the groups possible in category (A).

(B). Let us now take up the case where the transformations of the abridged group $U_1,\ U_2,\ U_3;\ V_1,\ V_2,\ V_3$ occur in our sought groups.

(1). Suppose no transformation $\Sigma\delta T$ occur free. Then we have

$$U_1+\sum{}^{i}\alpha_iT_i,\ U_2+\sum{}^{j}\beta_jT_j,\ U_3+\sum{}^{k}\gamma_kT_k,$$

and

$$V_1+\sum{}^{i}a_iT_i,\ V_2+\sum{}^{j}b_jT_j,\ V_3+\sum{}^{k}c_kT_k.$$

We easily find that this gives the group

$$xq,\ xp-yq,\ yp;\ r,\ xp+yq+2zr,\ (xz-y)p+yzq+z^2r.$$

The transformation $xp+yq$ may occur free, which gives the group

$$xq,\ xp,\ yq,\ yp;\ r,\ xp+yq+2zr,\ (xz-y)p+yzq+z^2r.$$

These groups leave two different straight lines in R_3 invariant.

(2). If one transformation $\sum{}^{\mu}\delta_\mu T_\mu$ occurs free, it is easy to see that *all* the T must then occur free. Thus we find the group

$$xq,\ xp-yq,\ yp;\ r,\ xp+yq+2zr,\ xzp+yzq+z^2r-yp,\quad p,\ q,\ zp,\ zq.$$

As usual, $xp+yq$ may occur free; and we find so the original G_{11}, of which the last is a subgroup which leaves a straight line invariant.

These are *all* of our required groups.

Lie has given general formulae for writing projective transformations in $(n-1)$ variables linear and homogeneous in n variables. For writing projective transformations in xyz linear and homogeneous in $x_1x_2x_3x_4$, the formulae are

$$xq\equiv x_1p_2,\ xr\equiv x_1p_3,\ yp\equiv x_2p_1,\ yr\equiv x_2p_3,$$
$$zp\equiv x_3p_1,\ zq\equiv x_3p_2,$$
$$p\equiv x_4p_1,\quad q\equiv x_4p_2,\quad r\equiv x_4p_3,$$
$$x.U\equiv -x_1p_4,\ y.U\equiv -x_2p_4,\ z.U\equiv -x_3p_4,$$
$$U+xp\equiv x_1p_1-x_4p_4,\ U+yq\equiv x_2p_2-x_4p_4,$$
$$U+zr\equiv x_3p_3-x_4p_4,$$
$$xp\equiv x_1p_1-\frac{1}{4}\sum_{1}^{4}{}^{i}x_ip_i,\ yq\equiv x_2p_2-\frac{1}{4}\sum_{1}^{4}{}^{i}x_ip_i,$$
$$zr\equiv x_3p_3-\sum_{1}^{4}{}^{i}x_ip_i,$$

where we write for brevity

$$U \equiv xp + yq + zr.$$

If we reverse these formulae of transformation, and write the transformation

$$\overline{U} \equiv \sum_1^4 {}_i\, x_i p_i$$

in the variables xyz, we get *zero, i. e.* the "*identical*" transformation. Thus we see that when we write our projective G_r, in xyz, linear and homogeneous in $x_1x_2x_3x_4$, the transformation $\overline{U}$ may also occur in the linear homogeneous group, free, or added with a constant coefficient to the other transformations.

CHAPTER I.

A Twisted Curve IIIO is Invariant in R_3.

§1.

The projective group written in the variables xyz has the form

$$p + 2xq + 3yr, \; xp + 2yq + 3zr, \; 3x^2p + 3xyq + 3xzr - 2yp - zq,$$

and leaves the curve *IIIO*

$$y - x^2 = 0, \; z - x^3 = 0,$$

invariant.

When written linear and homogeneous in $x_1x_2x_3x_4$ according to the formulae p. 310, the group has the form

$$x_4p_1 + 2x_1p_2 + 3x_2p_3, \; x_1p_1 - x_2p_2 + 3\,(x_4p_4 - x_3p_3), \; 2x_2p_1 + x_3p_2 + 3x_1p_4,$$

to which may be added

$$U \equiv \sum_1^4 {}_i\, x_i p_i.$$

In the following we shall only write the initial terms of our transformations, for the sake of brevity.

When we have found the transformations of a group in *finite* form we shall enclose them in a frame to catch the eye.

§2.

Thus the transformations IO of the groups we now seek have the forms—writing only the initial terms—

$$\begin{aligned} S_1 &\equiv x_4p_1 + 2x_1p_2 + 3x_2p_3, \\ S_2 &\equiv x_1p_1 - x_2p_2 + 3(x_4p_4 - x_3p_3), \\ S_3 &\equiv 2x_2p_1 + x_3p_2 + 3x_1p_4, \end{aligned}$$

to which may be added

$$U \equiv \sum_1^4 {}_i\, x_ip_i.$$

For the initial terms of these transformations the relations hold

$$(S_1S_2) = -2S_1,\ (S_1S_3) = S_2,\ (S_2S_3) = -2S_3,\ (S_iU) = 0.$$

Thus we see that if the transformation U does not occur free, it cannot occur at all.

We wish to see what transformations of an order higher than the first can occur.

If a transformation IO occurs it must have the form

$$Xf \equiv \xi_1(x_1x_2x_4)p_1 + \xi_2(x_1x_2x_3)p_2 + \xi_3(x_2x_3)p_3 + \xi_4(x_1x_4)p_4.$$

For we know that the four transformations zero $O:p_k$ must occur; thus, if ξ_1 were a function of x_3, we could by a proper combination of Xf with the p_k get a transformation IO of the form

$$Yf \equiv x_3p_1 + \eta_2p_2 + \eta_3p_3 + \eta_4p_4.$$

But Yf must be a transformation of our group, and so, linear in the S_i. We see at once that this is impossible, hence ξ_1 is free of x_3. Same way we see that ξ_2, ξ_3, ξ_4 have the above forms.

Since the ξ_k are analytical functions of their arguments, we may write

$$\begin{aligned} Xf \equiv (a_1x_1^2 + b_1x_2^2 + c_1x_4^2 + d_1x_1x_2 + e_1x_1x_4 + f_1x_2x_4)p_1 + (a_3x_2^2 + b_3x_2x_3 + c_3x_3^2)p_3 \\ + (a_2x_1^2 + b_2x_2^2 + c_2x_3^2 + d_2x_1x_2 + e_2x_1x_3 + f_2x_2x_3)p_2 + (a_4x_1^2 + b_4x_1x_4 + c_4x_4^2)p_4 \end{aligned}$$

where a, b, ... c_4 are certain constants.

Combine now Xf with p_3, and we get a transformation IO,

$$Z_1f \equiv (2c_2x_3 + e_2x_1 + f_2x_2)p_2 + (b_3x_2 + 2c_3x_3)p_3,$$

which must belong to our group. Thus we must have

$$Z_1f \equiv \sum_1^3 {}_k\, \alpha_kS_k + \alpha U,$$

where the α are certain constants. But we see at once that this relation gives

$$c_2 = e_2 = f_2 = b_3 = 2c_3 = 0.$$

Hence

$$Xf \equiv (a_1x_1^2 + b_1x_2^2 + c_1x_4^2 + d_1x_1x_2 + e_1x_1x_4 + f_1x_2x_4)\,p_1 + a_3x_2^2\,p_3 + (a_2x_1^2 + b_2x_2^2 + d_2x_1x_2)\,p_2 + (a_4x_1^2 + b_4x_1x_4 + c_4x_4^2)\,p_4.$$

Now combine Xf with p_4, and we find a transformation IO,

$$Z_2f \equiv (2c_1x_4 + e_1x_1 + f_1x_2)\,p_1 + (b_4x_1 + 2c_4x_4)\,p_4,$$

which must belong to our group. Thus as before we must have

$$Z_2f \equiv \sum_1^3{}^i \beta_i S_i + \beta U,$$

where the β are certain constants. We find without difficulty that this gives

$$c_1 = e_1 = f_1 = b_4 = c_4 = 0.$$

Proceeding in exactly the same manner with Xf and p_1 and p_2 we find

$$a_1 = d_1 = a_2 = d_2 = a_4 = b_1 = b_2 = a_3 = 0.$$

Thus we find that all constants are zero; that is, that no transformation IIO can occur at all. Also none of $IIIO$ can occur, for a transformation $IIIO$ combined with a transformation zero O gives a transformation IIO, which cannot occur. We see in this manner that *no transformation of an order higher than the first can occur.*

We have now two cases according as the transformation U occurs or not.

§3.

Suppose the transformation U occurs. In this case the transformations of our group are

$$p_1,\, p_2,\, p_3,\, p_4;\quad S_1 \equiv x_4p_1 + 2x_1p_2 + 3x_2p_3,\quad U \equiv \sum_1^4{}^i x_ip_i,$$
$$S_2 \equiv x_1p_1 - x_2p_2 + 3\,(x_4p_4 - x_3p_3);$$
$$S_3 \equiv 2x_2p_1 + x_3p_2 + 3x_1p_4.$$

Let us find the composition of our group. We see that the S_i and U are connected by normal relations, and so are the S_i among each other. We shall first choose the transformations zero O so that they are connected with U by normal relations, *i. e.* we shall "normalize" with U, and then we shall see how the transformations zero O are connected with the other transformations IO and with each other.

If now we combine p_1 and U we must get a transformation of our group, thus,

$$(p_1U) = p_1 + \sum_1^3{}^i \alpha_i S_i + \beta U, \quad (\alpha_i, \beta \text{ consts.}).$$

Let us introduce a new p_1 by means of the substitution

$$\bar{p}_1 \equiv p_1 + \sum_1^3{}^i A_i S_i + BU,$$

where A_i and B are certain constants. Then we have

$$(\bar{p}_1 U) \equiv \bar{p}_1 - \sum_1^3{}^i A_i S_i - BU + \sum_1^3{}^i \alpha_i S_i + \beta U.$$

Now put $$A_i = \alpha_i, \quad B = \beta,$$
and we have $$(\bar{p}_1 U) = \bar{p}_1.$$
Since nothing depends upon the symbol we use, we may write this

$$(p_1 U) = p_1,$$

remembering that this is a new p_1.

Proceeding in just the same manner we may write

$$\begin{cases} (p_2 U) = p_2, \\ (p_3 U) = p_3, \\ (p_4 U) = p_4. \end{cases}$$

Thus we have normalized the transformations zero O with U. Let us find the relations between the p_k and the S_i. We have

$$(p_1 S_1) = 2p_2 + \sum_1^3{}^i a_i S_i + bU.$$

Form Jacobi's Identity with p_1, S_1 and U, thus

$$((p_1 S_1)\, U) + ((S_1 U) p_1) + ((U p_1)\, S_1) = 0,$$

or $$2\,(p_2 U) - (p_1 S_1) = 0,$$

and $$(p_1 S_1) = 2p_2.$$

Thus p_1 and S_1 are connected by a normal relation. In exactly the same manner we find that the other transformations zero O and IO are connected by normal relations.

It remains to see how the transformations zero O are connected among each other.

We cannot say *a priori* of what order a transformation will be which we obtain by combining two transformations of zero O, since no transformations of *minus first order* can occur.

Thus we must write

$$(p_1p_2)=\sum_{i=1}^{4} a_ip_i+\sum_{k=1}^{3} b_kS_k+cU,$$

where a_i, b_k, c are certain constants.

Now form Jacobi's Identity with p_1, p_2 and U, thus

$$((p_1p_2)\,U)+((p_2U)\,p_1)+((Up_1)\,p_2)=0,$$

or

$$\sum_{i=1}^{4} a_ip_i-2\,(p_1p_2)=0,$$

i. e.

$$(p_1p_2)=0.$$

In the same way we find

$$(p_ip_k)=0.$$

Thus we have found the composition of our group, and find that all transformations are connected by normal relations.

A group of this composition whose transformations have the same initial terms is

$$\boxed{\begin{gathered}p_1,\ p_2,\ p_3,\ p_4;\ x_1p_1-x_2p_2+3\,(x_4p_4-x_3p_3)\,;\ 2x_2p_1+x_3p_2+3x_1p_4,\\ x_4p_1+2x_1p_2+3x_2p_3,\ x_1p_1+x_2p_2+x_3p_3+x_4p_4.\end{gathered}}$$

By Lie's theorem, p. 303, this is thus the group we seek. This group is *primitive*, for with any ordinary point which we hold is, as is easy to see, no plane pencil of directions invariantly connected, and also no direction.

§4.

Suppose U does not occur.

Here we have the four transformations zero O and the transformations IO,

$$\begin{aligned}S_1&=x_4p_1+2x_1p_2+3x_2p_3,\\ S_2&=x_1p_1-x_2p_2+3\,(x_4p_4-x_3p_3),\\ S_3&=2x_2p_1+x_3p_2+3x_1p_4.\end{aligned}$$

We see that the transformations IO are connected by normal relations. To find the composition of our group, let us normalize with S_2. We have

$$(p_1 S_2) = p_1 + \sum_1^3{}^i \alpha_i S_i, \quad (\alpha_i \text{ const.}),$$

and when we introduce a new p_1 by means of

$$\bar{p}_1 \equiv p_1 + \sum_1^3{}^i A_i S_i, \quad (A_i \text{ const.}),$$

we get

$$(\bar{p}_1 S_2) = \bar{p}_1 - \sum_1^3{}^i A_i S_i + \sum_1^3{}^i \alpha_i S_i - 2A_1 S_1 + 2A_3 S_3.$$

We can evidently choose our constants A_i so that

$$(\bar{p}_1 S_2) = \bar{p}_1,$$

or, as we may write it,

$$(p_1 S_2) = p_1.$$

In the same way we find

$$\begin{cases} (p_2 S_2) = -p_2, \\ (p_3 S_2) = -3p_3, \\ (p_4 S_2) = 3p_4. \end{cases}$$

Now we wish to see how the p_k are connected with S_1 and S_3. We have

$$(p_1 S_1) = 2p_2 + \sum_1^3{}^i a_i S_i, \quad (a_i \text{ const.}).$$

Form Jacobi's Identity with p_1, S_1 and S_2, thus

$$((p_1 S_1) S_2) + ((S_1 S_2) p_1) + ((S_2 p_1) S_1) = 0,$$

or

$$(p_1 S_1) = 2p_2.$$

Thus we see p_1 and S_1 are connected by a normal relation, and in same manner we find that all $(p_k S_i)$ are normal.

It only remains to see how the transformations zero O are connected among each other. We have

$$(p_1 p_2) = \sum_1^4{}^i \alpha_i p_i + \sum_1^3{}^k \beta_k S_k, \quad (\alpha, \beta \text{ const.}).$$

Form Jacobi's Identity with p_1, p_2 and S_2; thus

$$((p_1 p_2) S_2) - (p_1 p_2) + (p_1 p_2) = 0,$$

or

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \beta_1 = \beta_3 = 0.$$

Thus, $$(p_1p_2) = \beta_2 S_2.$$

Same way $$\left\{\begin{array}{l}(p_1p_3) = \gamma_1 S_1,\\ (p_1p_4) = (p_2p_3) = 0,\\ (p_2p_4) = \delta_3 S_3,\\ (p_3p_4) = \varepsilon_2 S_2.\end{array}\right.$$

Form Jacobi's Identity with p_1, p_2 and p_3; thus

$$((p_1p_2)p_3) + ((p_2p_3)p_1) + ((p_3p_1)p_2) = 0,$$

or $$3\beta_2 p_3 + 3\gamma_1 p_3 = 0, \quad \gamma_1 = -\beta_2.$$

Also $$((p_1p_2)S_1) - 3(p_1p_3) = 0,$$

or $$2\beta_2 S_1 - 3\gamma_1 S_1 = 0, \quad i.\,e.\ \beta_2 = \gamma_1 = 0.$$

Similarly we find without difficulty by means of Jacobi's Identity

$$\delta_3 = \varepsilon_2 = 0.$$

Hence all $$(p_i p_k) = 0,$$

and our transformations are connected by normal relations. Our group is evidently

$$\boxed{\begin{array}{c} p_1,\ p_2,\ p_3,\ p_4;\ x_4p_1 + 2x_1p_2 + 3x_2p_3,\ 2x_2p_1 + x_3p_2 + 3x_1p_4;\\ x_1p_1 - x_2p_2 + 3(x_4p_4 - x_3p_3).\end{array}}$$

This group is a subgroup of the one found in §3, and we see at once that this one is also *primitive* for the same reason that that one is.

CHAPTER II.

A Linear Complex is Invariant in R_3.

§1.

The projective group written in xyz has the form

$$p - yr,\ q + xr,\ r,\ xq,\ yp,\ xp - yq,\ xp + yq + 2zr,$$
$$zq + x.U,\ zp - y.U,\ xzp + yzq + z^2r,\ (\text{where } U \equiv xp + yq + zr),$$

and leaves the linear complex

$$dz + xdy - ydx = 0$$

invariant.

When written linear and homogeneous in $x_1x_2x_3x_4$ according to the formulae p. 310, the group has the form

$$x_4p_1 - x_2p_3,\ x_4p_2 + x_1p_3,\ x_4p_3,\ x_1p_2,\ x_1p_1 - x_2p_2,\ x_2p_1,$$
$$x_3p_1 + x_2p_4,\ x_3p_3 - x_4p_4,\ x_3p_2 - x_1p_4,\ x_3p_4,$$

to which may be added

$$U \equiv \sum_1^4{}^i x_ip_i.$$

§2.

Thus the transformations IO of the groups we now seek have the forms

$$S_1 \equiv x_4p_1 - x_2p_3,\ S_2 \equiv x_4p_2 + x_1p_3,\ S_3 \equiv x_4p_3,\ S_4 \equiv x_1p_2,$$
$$S_5 \equiv x_1p_1 - x_2p_2,\ S_6 \equiv x_2p_1,\ S_7 \equiv x_3p_1 + x_2p_4,\ S_8 \equiv x_3p_3 - x_4p_4,$$
$$S_9 \equiv x_3p_2 - x_1p_4,\ S_{10} \equiv x_3p_4,$$

to which may be added

$$U \equiv \sum_1^4{}^i x_ip_i.$$

Let us see what transformations of an order higher than the first can occur.

Let s be the maximum order, so that we suppose a transformation of form

$$Xf \equiv \xi_1^{(s)}p_1 + \xi_2^{(s)}p_2 + \xi_3^{(s)}p_3 + \xi_4^{(s)}p_4$$

occurs, where the index (s) shows that the ξ_k are of the s^{th} degree.

Since in our group x_1, x_2 are equally privileged with x_3, x_4, we can suppose that $\xi_1^{(s)}$ and $\xi_2^{(s)}$ are not both zero. If now $\xi_1^{(s)} \neq 0$, by combining Xf with

$$x_1p_2,\ x_1p_3 - x_4p_2,\ x_1p_4 - x_3p_2$$

in proper order, we see that

$$Yf \equiv x_1^sp_1 + \bar{\xi}_2^{(s)}p_2 + \bar{\xi}_3^{(s)}p_3 + \bar{\xi}_4^{(s)}p_4$$

occurs. Combine p_1 and Yf, hence

$$Zf \equiv sx_1^{s-1}p_1 + y_2^{(s-1)}p_2 + y_3^{(s-1)}p_3 + y_4^{(s-1)}p_4$$

must be a transformation of our group.

Combine now Zf and Yf, hence

$$-sx_1^{2s-2}p_1 + \xi_2^{(2s-2)}p_2 + \xi_3^{(2s-2)}p_3 + \xi_4^{2s-2}p_4$$

must occur. But s is the maximum order, hence

$$2s - 2 < s + 1, \text{ or } s < 3.$$

Thus when $\xi_1^{(s)} \neq 0$, $s < 3$.

If $\xi_1^{(s)} = 0$, then $\xi_2^{(s)} \neq 0$, and

$$Xf \equiv \xi_2^{(s)}p_2 + \xi_3^{(s)}p_3 + \xi_4^{(s)}p_4.$$

Combine this transformation with

$$x_2p_1,\ x_2p_3 - x_4p_1,\ x_2p_4 + x_3p_1$$

in proper order, and we find that

$$x_2^s p_2 + \bar{\bar{\xi}}_1^{(s)}p_1 + \bar{\bar{\xi}}_3^{(s)}p_3 + \bar{\bar{\xi}}_4^{(s)}p_4$$

must occur, and proceeding just as with Yf above, we see $s < 3$.

Thus at all events is $s < 3$.

Now let us see what transformations *IIO*, if any, can occur. If any transformation *IIO* occurs, we wish now to show that one of the form

$$\bar{\xi}_2^{(2)}p_2 + \bar{\xi}_3^{(2)}p_3 + \bar{\xi}_4^{(2)}p_4$$

must occur, where $\bar{\xi}_2^{(2)} \neq 0$.

First we must prove that no transformation *IIO* of the form

$$Xf \equiv \xi_1p_1 + \xi_2p_2$$

can occur, and since x_1, x_2 are equally privileged with x_3, x_4, of course then none of the form

$$\xi_3p_3 + \xi_4p_4$$

could occur.

If Xf occurs, then must ξ_1 be $\neq 0$. For if not, a transformation

$$\xi_2p_2 \equiv x_1^2p_2$$

must occur. Combine this transformation with x_2p_1 and we see

$$2x_1x_2p_2 - x_1^2p_1$$

must occur. Combine the last transformation with $x_1^2p_2$ and we get a transformation *IIIO*, which is impossible.

Thus $\xi_1 \neq 0$ when Xf occurs.

By combining Xf properly with x_3p_4 and x_1p_2, we can arrange so that

$$\bar{\xi}_1(x_1x_3)\,p_1 + \bar{\xi}_2p_2$$

must occur. But from the transformations *IO* we see that ξ_1 can contain no x_3 here. Thus a transformation

$$\overline{X}f \equiv x_1^2p_1 + \bar{\xi}_2p_2$$

occurs. Combine $x_1p_1 - x_2p_2$ with $\overline{X}f$ and we see

$$Yf \equiv x_1^2p_1 + \left\{x_1\frac{\partial\xi_2}{\partial x_1} - x_2\frac{\partial\xi_2}{\partial x_2} + \xi_2\right\}p_2$$

must be a transformation of our group. Here Yf must be the same transformation as $\overline{X}f$, otherwise, by subtracting, we would find a transformation

$$\eta_2 p_2$$

which would belong to our group. But we saw no such transformation can occur; thus

$$Yf \equiv \overline{X}f,$$

and hence

$$x_1 \frac{\partial \xi_2}{\partial x_1} + x_2 \frac{\partial \xi_2}{\partial x_2} = 0, \text{ or } \xi_2 = \xi_2(x_1 x_2,\ x_3 x_4).$$

Hence

$$\overline{X}f \equiv x_1^2 p_1 + (a x_1 x_2 + b x_3^2 + c x_3 x_4 + d x_4^2) p_2.$$

Combine this transformation with p_3, p_4 and we see

$$b = c = d = 0;$$

thus

$$x_1^2 p_1 + a x_1 x_2 p_2$$

occurs. Combine this with p_1 and we see at once

$$a = -2.$$

Combine now

$$x_1^2 p_1 - 2 x_1 x_2 p_2$$

with $x_1 p_2$ and we find that

$$x_1^2 p_2$$

must occur. But we saw that this was impossible; thus, no transformation of form

$$\xi_1^{(2)} p_1 + \xi_2^{(2)} p_2 \text{ or } \xi_3^{(2)} p_3 + \xi_4^{(2)} p_4$$

can occur. Thus if a transformation of form

$$\xi_1^{(2)} p_1 + \xi_2^{(2)} p_2 + \xi_3^{(2)} p_3 + \xi_4^{(2)} p_4$$

occurs, but not

$$Zf \equiv \eta_2^{(2)} p_2 + \eta_3^{(2)} p_3 + \eta_4^{(2)} p_4,$$

we must suppose that $\xi_1^{(2)} \neq 0$. Hence

$$Xf \equiv x_1^2 p_1 + \xi_2^{(2)} p_2 + \xi_3^{(2)} p_3 + \xi_4^{(2)} p_4$$

must occur. Combine this transformation with $x_1 p_1 - x_2 p_2$ and we see

$$Yf \equiv x_1^2 p_1 + \left(x_1 \frac{\partial \xi_2}{\partial x_1} - x_2 \frac{\partial \xi_2}{\partial x_2} + \xi_2\right) p_2 + \left(x_1 \frac{\partial \xi_3}{\partial x_1} - x_2 \frac{\partial \xi_3}{\partial x_2}\right) p_3 + (\ ) p_4$$

must belong to our group. Subtract Xf from Yf and we see that

$$\left(x_1 \frac{\partial \xi_2}{\partial x_1} - x_2 \frac{\partial \xi_2}{\partial x_2}\right) p_2 + \left(x_1 \frac{\partial \xi_3}{\partial x_1} - x_2 \frac{\partial \xi_3}{\partial x_2} - \xi_3\right) p_3 + (\ ) p_4$$

also belongs to the group. But this is a transformation of the form Zf, and since by hypothesis none such can occur, we must have

$$x_1 \frac{\partial \xi_2}{\partial x_1} - x_2 \frac{\partial \xi_2}{\partial x_2} = 0, \; x_1 \frac{\partial \xi_3}{\partial x_1} - x_2 \frac{\partial \xi_3}{\partial x_2} = \xi_3, \; x_1 \frac{\partial \xi_4}{\partial x_1} - x_2 \frac{\partial \xi_4}{\partial x_2} = \xi_4,$$

and hence $\xi_2 = \xi_2(x_1x_2, x_3, x_4), \; \xi_3 = \xi_3(x_1x_2, x_3, x_4) . x_1$
and $\xi_4 = \xi_4(x_1x_2, x_3, x_4) . x_1.$

Since our transformation is of the *IIO*,

$$\xi_3 = x_1 . \xi_3(x_3, x_4), \; \xi_4 = x_1 . \xi_4(x_3, x_4).$$

Thus a transformation of the form

$$x_1^2 p_1 + (ax_1x_2 + bx_3x_4 + cx_3^2 + dx_4^2) p_2 + x_1(ex_3 + gx_4) p_3 + x_1(e'x_3 + g'x_4) p_4$$

occurs. By combining this transformation p_1, p_2, p_3, p_4 we easily see that also

$$x_1^2 p_1 - 2x_1x_2p_2 + (ex_3x_4 + cx_3^2 + dx_4^2) p_2 + x_1(ex_3 + 2dx_4) p_3 + x_1(2cx_3 + ex_4) p_4$$

must belong to our group. Combine this transformation with x_1p_2 and we see that $x_1^2 p_2$ must occur, which we saw to be impossible.

Thus a transformation of the form

$$Zf = \xi_2^{(2)} p_2 + \xi_3^{(2)} p_3 + \xi_4^{(2)} p_4$$

must occur, if any transformation *IIO* occur at all; and here $\xi_2^{(2)} \neq 0$, and not $\xi_3^{(2)}, \xi_4^{(2)}$ both can be zero.

Combine properly with x_1p_2 and x_3p_4, and we see that a transformation

$$Xf \equiv \xi_2(x_1, x_3) p_2 + \xi_3 p_3 + \xi_4 p_4$$

occurs, where all ξ_k are free of x_2. Combine Xf again with x_3p_4 and we find

$$x_3\left(\frac{\partial \xi_3}{\partial x_4} p_3 + \frac{\partial \xi_4}{\partial x_4} . p_4\right) - \xi_3 p_4.$$

But no such transformation can occur, hence

$$\xi_3 = \xi_3(x_1x_3), \; \xi_4 = \frac{x_4}{x_3} . \xi_3 + \eta_4(x_1x_3).$$

From the transformation *IO* we see that no x_1 can occur in ξ_3 here; thus

$$\xi_3 = ax_3^2, \; \xi_4 = ax_3x_4 + \eta_4(x_1x_3), \qquad (a = \text{const.}).$$

Thus a transformation *IIO* of form

$$\xi_2(x_1x_3) p_2 + ax_3^2 p_3 + \{ax_3x_4 + \eta_4(x_1x_3)\} p_4$$

must belong to our group. Combine this with p_3 and we see $a = 0$; thus

$$Xf = \xi_2(x_1x_3) p_2 + \xi_4(x_1x_3) p_4,$$

where $\xi_2 . \xi_4 \neq 0$. By combining Xf properly with $x_1 p_3 - x_4 p_2$ we see that

$$x_1^2 p_4 + \bar{\xi}_2 (x_1 x_3) p_2$$

occurs, or

$$x_1^2 p_4 + (d x_1^2 + b x_1 x_3 + c x_3^2) p_2 .$$

Combine with p_1 and p_3 and we see

$$b = -2, \quad c = 0;$$

thus

$$x_1^2 p_4 + (d x_1^2 - 2 x_1 x_3) p_2$$

occurs. Combine with $x_4 p_3$ and we see

$$X_1 f = x_1^2 p_3 + 2 x_1 x_4 p_2$$

occurs. Combine $x_2 p_1$ with $X_1 f$ and we see

$$X_2 f \equiv x_1 x_4 p_1 - x_2 x_4 p_2 - x_1 x_2 p_3$$

occurs. Combine $X_1 f$ and $X_2 f$ and we get a transformation *IIIO*; but such a transformation cannot occur; *thus no transformation IIO can occur.*

We now have two cases according as U occurs or not. We easily see that U cannot occur additively.

§3.

Suppose U occurs free.

In this case we "normalize" with U just as in §3, Chap. I, and find without any difficulty the group

$$\boxed{\begin{array}{l} p_1, p_2, p_3, p_4;\ x_4 p_1 - x_2 p_3,\ x_4 p_2 + x_1 p_3,\ x_4 p_3,\ x_1 p_1 - x_2 p_2,\ x_2 p_1,\ x_1 p_2, \\ x_3 p_1 + x_2 p_4,\ x_3 p_2 - x_1 p_4,\ x_3 p_4,\ x_3 p_3 - x_4 p_4,\ x_1 p_1 + x_2 p_2 + x_3 p_3 + x_4 p_4 . \end{array}}$$

We see at once that this group is *primitive*, for the same reason as those found in the last chapter are primitive.

§4.

Suppose U does not occur free.

Our transformations *IO* are

$$\begin{gathered} x_4 p_1 - x_2 p_3,\ x_4 p_2 + x_1 p_3,\ x_4 p_3,\ x_1 p_2,\ x_1 p_1 - x_2 p_2,\ x_2 p_1, \\ x_3 p_1 + x_2 p_4,\ x_3 p_2 - x_1 p_4,\ x_3 p_4,\ x_3 p_3 - x_4 p_4 . \end{gathered}$$

Let us designate the first nine of these transformations in the order written respectively by $S_1 \ldots . S_9$, and put

$$T = x_1 p_1 - x_2 p_2 + (x_3 p_3 - x_4 p_4) .$$

These transformations IO are connected by normal relations, and we see by combination that U cannot occur additively. We may at once choose the p_k, without loss of generality, so that

$$\begin{cases} (p_1T) = p_1, \\ (p_2T) = -p_2, \\ (p_3T) = p_3, \\ (p_4T) = -p_4. \end{cases}$$

Further, we find without difficulty by means of Jacobi's Identity that now all the relations (p_kS_i) are normal. It remains to see how the transformations zero O are connected among each other. We have

$$(p_1p_2) = \sum_1^4{}^i b_ip_i + \sum_1^9{}^k a_kS_k + bT.$$

Form Jacobi's Identity with p_1, p_2 and T, thus

$$((p_1p_2)\,T) = 0,$$

or $$(p_1p_2) = a_2S_2 + a_5S_5 + a_7S_7 + bT.$$

Also now $$((p_1p_2)\,S_3) = 0, \quad i.\,e.\ \ a_7 = b = 0.$$

Finally $$((p_1p_2)\,S_6) = 0, \quad i.\,e.\ \ a_2 = a_5 = 0.$$

Thus $$(p_1p_2) = 0.$$

Analogously we find $$(p_ip_k) = 0.$$

Thus we find that all of our transformations are connected by normal relations. Our group is evidently

$$\boxed{\begin{array}{c} p_1, p_2, p_3, p_4;\ x_1p_1 - x_2p_3,\ x_4p_2 + x_1p_3,\ x_4p_3,\ x_1p_2,\ x_1p_1 - x_2p_2; \\ x_2p_1,\ x_3p_1 + x_2p_4,\ x_3p_2 - x_1p_4,\ x_3p_4,\ x_3p_3 - x_4p_4. \end{array}}$$

This is a subgroup of the one in §3, and is, for the same reason as that one, *primitive*.

CHAPTER III.

A Surface of the Second Degree and One of its Generators Invariant in R_3.

§1.

The projective group written in the variables xyz has the form

$$r + xq,\ yq + zr,\ (xz - y)\,p + yzq + z^2r,\ xp + yq,\ p + zq,$$

and leaves the surface of the second degree

$$y - zx = 0$$

with one of its generators, invariant.

When written linear and homogeneous in $x_1x_2x_3x_4$ according to the formulae p. 310, our group has the form*

$$x_4p_1 - x_2p_3,\ x_1p_1 - x_2p_2 + x_3p_3 - x_4p_4,\ x_3p_2 - x_1p_4,\ x_3p_1 - x_2p_4,$$
$$x_1p_1 - x_2p_2 - x_3p_3 + x_4p_4,$$

to which may be added

$$U \equiv \sum_1^4{}_i\, x_ip_i.$$

§2.

Thus the transformations IO of the groups we now seek have the forms

$$S_1 \equiv x_4p_1 - x_2p_3,\ S_2 = x_1p_1 - x_2p_2 + x_3p_3 - x_4p_4,\ S_3 = x_3p_2 - x_1p_4,$$
$$S_4 = x_3p_1 - x_2p_4,\ S_5 = x_1p_1 - x_2p_2 - x_3p_3 + x_4p_4,$$

to which may be added

$$U \equiv \sum_1^4{}_i\, x_ip_i.$$

For the initial terms of these transformations the relations hold,

$$(S_1S_2) = 2S_1,\ (S_1S_3) = S_2,\ (S_2S_3) = 2S_3,\ (S_4S_5) = 2S_4,$$
$$(S_1S_4) = (S_1S_5) = (S_2S_4) = (S_2S_5) = (S_3S_4) = (S_3S_5) = 0,$$
$$(S_iU) = 0.$$

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* The form in the text is not that given immediately by the formulae p. *cit.*, but an equivalent one which we choose on account of the following calculations being so somewhat more convenient. We obtain the group given by the formulae p. *cit.*, when we perform the simple transformation

$$x_1' = x_3,\ x_2' = -x_1,\ x_3' = x_2,\ x_4' = x_4,$$

on the group in the text.

The same change is made in the next three chapters.

Thus we see that if U does not occur free, it can occur additively only with S_5, in form

$$S_5 + aU, \qquad (a \text{ const.}).$$

Let us see what transformations of an order higher than the first can occur.

No transformation IIO of the form

$$\xi_1^{(2)} p_1 + \xi_4^{(2)} p_4$$

can occur; for then we would have

$$\xi_1^{(2)} \equiv \xi_1^{(2)}(x_1 x_3), \quad \xi_4^{(2)} \equiv \xi_4^{(2)}(x_2 x_4),$$

and by combining with p_k, we find that then

$$\xi_1^{(2)} \equiv \xi_4^{(2)} \equiv 0.$$

Thus if a transformation IIO of form

$$Xf \equiv \sum_1^4{}_i\, \xi_i^{(2)} p_i$$

occurs, we must have either $\xi_2^{(2)} \neq 0$, or $\xi_3^{(2)} \neq 0$. We see also

$$\xi_2^{(2)} \equiv \xi_2^{(2)}(x_2 x_3), \quad \xi_3^{(2)} \equiv \xi_3^{(2)}(x_2 x_3), \quad \xi_1^{(2)} \equiv \xi_1^{(2)}(x_1 x_3 x_4), \quad \xi_4^{(2)} \equiv \xi_4^{(2)}(x_1 x_2 x_4).$$

Thus if

$$Yf \equiv \sum_1^4{}_i\, \xi_i^{(s)} p_i$$

is a transformation sO which occurs, by combining Yf properly with $x_2 p_3 - x_4 p_1$, we can arrange so that

$$x_2^s p_2 + \eta_3^{(s)} p_3 + \eta_1^{(s)} p_1 + \eta_4^{(s)} p_4$$

must occur, when we suppose $\xi_2^{(s)} \neq 0$. In this case thus, when s is the maximum order, we must have $s < 3$.

If $\xi_2^{(s)} = 0$, then $\xi_3^{(s)} \neq 0$, and

$$x_3^s p_3 + \eta_2^{(s)} p_2 + \eta_1^{(s)} p_1 + \eta_4^{(s)} p_4$$

must occur. Thus, as above, $s < 3$.

Thus, at all events, no transformation of an order higher than the second can occur.

What transformations IIO can occur in our group?

(1). Suppose

$$\sum_1^4{}_i\, \xi_i^{(2)} p_i$$

is a transformation IIO which occurs, and let $\xi_2^{(2)}$ be $\neq 0$. Then as above a transformation

$$Xf \equiv x_2^2 p_2 + \xi_3(x_2 x_3) p_3 + \xi_1(x_1 x_3 x_4) p_1 + \xi_4(x_2 x_4) p_4$$

must occur. Combine Xf with $x_1 p_1 - x_2 p_2$, and we see

$$Yf \equiv x_2^2 p_2 + x_2 \frac{\partial \xi_3}{\partial x_2} p_3 - \left(x_1 \frac{\partial \xi_1}{\partial x_1} - \xi_1 \right) p_1 + x_2 \frac{\partial \xi_4}{\partial x_2} p_4$$

occurs. Either Yf is the same transformation as Xf, or not; if not, a transformation IIO of the form

$$\eta_3(x_2 x_3) p_3 + \eta_1(x_1 x_3 x_4) p_1 + \eta_4(x_2 x_4) p_4$$

which we get by subtracting Yf from Xf, must occur. Here we must have $\eta_3 \neq 0$, and by combining the last transformation properly with $x_2 p_3 - x_4 p_1$, we see a transformation of form

$$Zf \equiv x_2^2 p_3 + \bar{\eta}_1(x_1 x_3 x_4) p_1 + \bar{\eta}_4(x_2 x_4) p_4$$

occurs.

By combining p_3 with Zf we see that $\bar{\eta}_1 = \bar{\eta}_1(x_1 x_4)$.

Combine Zf with $x_3 p_3 - x_4 p_4$, and we see

$$x_2^2 p_3 + x_4 \frac{\partial \bar{\eta}_1}{\partial x_4} p_1 + \left(x_4 \frac{\partial \bar{\eta}_4}{\partial x_4} - \bar{\eta}_4 \right) p_4$$

must occur. Subtract this transformation from Zf and we find

$$x_4 \frac{\partial \bar{\eta}_1}{\partial x_4} = \bar{\eta}_1, \quad x_4 \frac{\partial \bar{\eta}_4}{\partial x_4} = 2\bar{\eta}_4,$$

or $\bar{\eta}_1 = \alpha_1 x_1 x_4, \quad \bar{\eta}_4 = \beta_4 x_4^2, \qquad (\alpha_1, \beta_4 \text{ const.}).$

Thus $Zf \equiv x_2^2 p_3 + \alpha_1 x_1 x_4 p_1 + \beta_4 x_4^2 p_4.$

By combining Zf with p_1 and p_4 we find

$$\alpha_1 = \beta_4 = 0.$$

Thus $Zf = x_2^2 p_3,$

and no such transformation can occur.

Thus we find that Yf must be the same transformation as Xf; that is,

$$x_2 \frac{\partial \xi_3}{\partial x_2} = \xi_3, \quad x_1 \frac{\partial \xi_1}{\partial x_1} = 0, \quad x_2 \frac{\partial \xi_4}{\partial x_2} = \xi_4.$$

Thus $\xi_3 = a x_2 x_3, \quad \xi_1 = \xi_1(x_3, x_4), \quad \xi_4 = b x_2 x_4,$

or $x_2^2 p_2 + a x_2 x_3 p_3 + (c x_3^2 + d x_3 x_4 + e x_4^2) p_1 + b x_2 x_4 p_4$

occurs.

By combining this transformation with p_3, p_4, p_2 respectively, and expressing the results linearly in the transformation IO, we find when U occurs free,

$$c = e = 0;\ a = b = 1 = -d.$$

Thus in this case the transformation IIO,

$$X_1f \equiv x_2^2p_2 + x_2x_3p_3 - x_3x_4p_1 + x_2x_4p_4$$

occurs. Combine this transformation with $x_3p_2 - x_1p_4$ and we find that

$$X_2f \equiv x_3^2p_3 + x_2x_3p_2 + x_1x_2p_1 - x_1x_2p_4$$

also occurs, and by combining the general transformation IIO with p_k we find that X_1f and X_2f are the only transformations IIO which can occur when U occurs free. We easily see that if U does not occur, no transformation IIO can occur at all, and if U occurs in the form

$$S_5 + \alpha U \qquad (\alpha \text{ const.})$$

we find that X_1f and X_2f can occur where

$$\alpha = -2.$$

Now we come to case

(2), where $\xi_2^{(2)} \equiv 0$ in the original transformation IIO,

$$Xf \equiv \sum_1^4{}_i\, \xi_i^{(2)}p_i.$$

In this case we have a transformation of form

$$\eta_3p_3 + \eta_4p_4 + \eta_1p_1,$$

and we saw on the last page that no such transformation could occur.

In finding our groups we shall now distinguish the two cases where transformations IIO occur, and where no transformations IIO occur.

§3.

Suppose no transformations IIO occur.

If the transformation U occurs free, we normalize as usual with it, and find without difficulty the group

$$\boxed{\begin{array}{l} p_1, p_2, p_3, p_4;\ x_4p_1 - x_2p_3,\ x_1p_1 - x_2p_2 + x_3p_3 - x_4p_4,\ x_3p_2 - x_1p_4, \\ x_3p_1 - x_2p_4,\ x_1p_1 - x_2p_2 - x_3p_3 + x_4p_4,\ x_1p_1 + x_2p_2 + x_3p_3 + x_4p_4. \end{array}}$$

But this group is imprimitive, since the family of ∞^2 manifoldnesses

$$\begin{cases} x_2 = \text{const.} \\ x_3 = \text{const.} \end{cases}$$

is invariant.

Suppose U does not occur free. Our transformations are then

$$p_k;\ x_4p_1 - x_2p_3 \equiv S_1,\ x_1p_1 - x_2p_2 + x_3p_3 - x_4p_4 \equiv S_2,\ x_3p_2 - x_1p_4 \equiv S_3,$$
$$x_3p_1 - x_2p_4 \equiv S_4,$$
$$S_5 + \alpha.\, U \equiv x_1p_1 - x_2p_2 - x_3p_3 + x_4p_4 + \alpha.\sum_1^4{}^i x_ip_i \equiv T,$$

where α is some constant.

Let us normalize with S_2.

We easily see that we can, without loss of generality, choose the transformations zero O, so that

$$\begin{cases} (p_1S_2) = p_1, \\ (p_2S_2) = -p_2, \\ (p_3S_2) = p_3, \\ (p_4S_2) = -p_4, \end{cases}$$

and we find without any difficulty, in the usual manner, by means of Jacobi's Identity, that all the relations (p_kS_i) and (p_kT) are normal. We wish to see how the transformations zero O are connected among each other.

We find directly from forming Jacobi's Identity with p_i, p_k and S_2,

$$\begin{cases} (p_1p_2) = b_2S_2 + bT + b_4S_4, \\ (p_1p_3) = b_1S_1, \\ (p_1p_4) = a_2S_2 + aT + a_4S_4, \\ (p_2p_3) = \beta_2S_2 + \beta T + \beta_4S_4, \\ (p_2p_4) = \gamma_3S_3, \\ (p_3p_4) = c_2S_2 + cT + c_4S_4, \end{cases}$$

where the $b_2b \ldots . c_4$ are certain constants.

Now form Jacobi's Identity with p_1, p_2, S_1, thus

$$((p_1p_2)\,S_1) + (p_1p_3) \equiv 0,$$

or

$$-2b_2S_1 + b_1S_1 \equiv 0,\ b_1 = 2b_2.$$

Proceed in exactly the same manner with the other p_i, p_k, S_3 and p_i, p_k, T, and we find a number of relations between our constants, viz:

$$a = b = c = a_4 = a_2 = \beta_2 = 0.$$
$$\alpha . b_1 = 0,\ \beta_4(\alpha - 2) = 0,\ \beta(\alpha - 1) = 0,\ c_4(\alpha - 1) = 0,\ b_4(\alpha - 1) = 0;$$
$$b_1 = \gamma_3 = 2b_2,\ b_4 = \beta = -c_4,\ b_2 = c_2.$$

(1). *Suppose* $\alpha = 1$, then $b_1 = \gamma_3 = b_2 = \beta_4 = 0$, and we have

$$(p_1p_2) = b_4S_4,\ (p_2p_3) = b_4T,\ (p_3p_4) = -b_4S_4,$$

and by forming Jacobi's Identity with p_1, p_2, p_3, we find $b_4 = 0$.

Hence all of our transformations are connected by normal relations in this case, and we find the group

$$\boxed{\begin{array}{c} p_1, p_2, p_3, p_4;\ x_4p_1 - x_2p_3,\ x_1p_1 - x_2p_2 + x_3p_3 - x_4p_4;\ x_3p_2 - x_1p_4, \\ x_3p_1 - x_2p_4,\ x_1p_1 - x_2p_2 - x_3p_3 + x_4p_4,\ x_1p_1 + x_4p_4. \end{array}}$$

This is a subgroup of the last one found, and is imprimitive for the same reason that that one is.

(2). *Suppose* $\alpha \neq 1$.

(a). *Let* $\alpha = 2$; hence

$$b_1 = \gamma_3 = b_2 = c_2 = c_4 = b_4 = \beta = 0.$$

All relations are normal except

$$(p_2p_3) = \beta_4S_4.$$

Thus we see that p_1, p_4 is an invariant G_2 in our group, and hence the group is imprimitive.*

(b). *Suppose* $(\alpha - 1)(\alpha - 2) \neq 0$; hence

$$\beta = c_4 = b_4 = \beta_4 = 0,$$

and we have

$$\begin{aligned} &(p_1p_2) = b_2S_2, \\ &(p_1p_3) = 2b_2S_1, \\ &(p_1p_4) = 0 = (p_2p_3), \\ &(p_2p_4) = 2b_2S_3, \\ &(p_3p_4) = b_2S_2. \end{aligned}$$

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* For
$$p_1 = 0,\ p_4 = 0$$
is an invariant complete system with the two solutions
$$x_2 = \text{const.},\ x_3 = \text{const.}$$
Thus our group leaves the family of $\infty^2 M_2$, $x_2 = \text{const.}$, $x_3 = \text{const.}$, invariant.

Form Jacobi's Identity with p_1, p_2, p_3 and we see at once

$$b_2 = 0.$$

Hence all of our relations are in this case normal, and we find again the imprimitive group written above.

§4.

Suppose transformations IIO occur.

Let us further suppose—

(a). *U occurs free.* In this case we normalize with U as usual and find without difficulty the group:

$$\boxed{\begin{array}{l} p_1, p_2, p_3, p_4; \; x_4p_1 - x_2p_3, \; x_1p_1 - x_2p_2 + x_3p_3 - x_4p_4, \; x_3p_2 - x_1p_4, \\ x_3p_1 - x_2p_4, \; x_1p_1 - x_2p_2 - x_3p_3 + x_4p_4, \; x_1p_1 + x_2p_2 + x_3p_3 + x_4p_4, \\ x_3 . U - (x_1x_2 + x_3x_4)p_4, \; x_2 . U - (x_1x_2 + x_3x_4)p_1. \end{array}}$$

This group is imprimitive, since the family of ∞^2 manifoldnesses,

$$\begin{cases} x_2 = \text{const.} \\ x_3 = \text{const.} \end{cases}$$

is invariant. Let us now suppose

(b). *U does not occur free.* Here, as we saw above,

$$S_5 - 2U$$

must occur. Thus our transformations have the form

$$\begin{aligned} &p_k, \; x_4p_1 - x_2p_3 \equiv S_1, \; x_1p_1 - x_2p_2 + x_3p_3 - x_4p_4 \equiv S_2, \\ &\quad x_3p_2 - x_1p_4 \equiv S_3, \; x_1p_1 - x_2p_2 - x_3p_3 + x_4p_4 \equiv S_4, \\ &T \equiv - S_5 + 2U \equiv x_1p_1 + 3\,(x_2p_2 + x_3p_3) + x_4p_4, \\ &\qquad X_1f \equiv x_3 . U - (x_1x_2 + x_3x_4)\,p_4, \; X_2f \equiv x_2 . U - (\;)\,p_1. \end{aligned}$$

Let us normalize with T. We have

$$(S_1T) = aX_1f + bX_2f, \qquad (a,\ b \text{ const.}).$$

Let us introduce a new S_1 by means of

$$\overline{S}_1 \equiv S_1 + AX_1f + BX_2f.$$

Then $$(\overline{S}_1T) = aX_1f + bX_2f - 3\,(AX_1f) + BX_2f).$$

Here we may choose our arbitrary constants A, B so that

$$(\overline{S}_1 T) = 0,$$

or

$$(S_1 T) = 0,$$

as we may write it.

Same way, we may without loss of generality choose the other S_k so that the relations $(S_k T)$ are all normal. The S_k and $X_i f$ are already connected by normal relations; we wish to find how the S_k are connected among each other. We have

$$(S_1 S_2) = 2S_1 + \alpha_1 X_1 f + \beta_1 X_2 f.$$

Form Jacobi's Identity with S_1, S_2 and T; thus

$$((S_1 S_2)\,T) + ((S_2 T)\,S_1) + ((T S_1)\,S_2) \equiv 0,$$

or

$$((S_1 S_2)\,T) \equiv 0,$$

i. e.

$$\alpha_1 = \beta_1 = 0.$$

Hence

$$(S_1 S_2) = 2S_1,$$

a normal relation. Analogously we find that all $(S_i S_k)$ are normal.

Now let us normalize the transformations zero O. We have

$$(p_1 T) = p_1 + \sum_1^4{}^i \alpha_i S_i + \alpha T + \beta X_1 f + \gamma X_2 f.$$

Introduce a new p_1 by means of

$$\overline{p}_1 \equiv p_1 + \sum_1^4{}^i A_i S_i + AT + \sum_1^2{}^k A'_k X_k f.$$

Thus

$$(\overline{p}_1 T) \equiv \overline{p}_1 - \sum_1^4{}^i A_i S_i - AT - \sum_1^2{}^k A_k X_k f + \sum_1^4{}^i \alpha_i S_i + \alpha T + \beta X_1 f$$
$$+ \gamma X_2 f - 2A_4 S_4 - 3\sum_1^2{}^k A'_k X_k f.$$

Here we may choose the constants so that

$$(\overline{p}_1 T) = \overline{p}_1,$$

or

$$(p_1 T) = p_1.$$

Analogously

$$\begin{cases} (p_2 T) = 3p_2, \\ (p_3 T) = 3p_3, \\ (p_4 T) = p_4. \end{cases}$$

Now to find how the p_k are connected with the other transformations of our group. We have

$$(p_1 X_1 f) \equiv S_4 + a_1 X_1 f + a_2 X_2 f.$$

Form Jacobi's Identity with p_1, $X_1 f$ and T; thus

$$((p_1 X_1 f)\, T) + ((X_1 f T) p_1) - ((p_1 T)\, X_1 f) \equiv 0,$$

or

$$-2S_4 - 3\,(a_1 X_1 f + a_2 X_2 f) + 2\,(p_1 X_1 f) = 0,$$

i. e.

$$(p_1 X_1 f) = S_4.$$

Analogously we find that the other $(p_k X_i f)$ are also normal.

Now for the $(p_i S_k)$. We have

$$(p_1 S_1) = \sum\limits_{1}^{4}{}^{i}\, \alpha_i S_i + \alpha T + \sum\limits_{1}^{2}{}^{k}\, \beta_k X_k f.$$

Form Jacobi's Identity with p_1, S_1, T, thus

$$((p_1 S_1)\, T) - (p_1 S_1) = 0,$$

i. e.

$$(p_1 S_1) = 0.$$

In the same manner we find that the other $(p_k S_i)$ are normal also.

It remains to see how the transformations zero O are connected among each other. We have

$$(p_1 p_2) = \sum\limits_{1}^{4}{}^{i}\, a_i p_i + \sum\limits_{1}^{4}{}^{k}\, b_k S_k + bT + \sum\limits_{1}^{2}{}^{j}\, c_j X_j f.$$

Form Jacobi's Identity with p_1, p_2, T, thus

$$((p_1 p_2)\, T) - 4\,(p_1 p_2) = 0,$$

i. e.

$$(p_1 p_2) = 0.$$

Analogously we find $(p_i p_k) = 0$.

Thus all of our transformations are connected by normal relations, and we find the group:

$$\boxed{\begin{array}{l} p_1,\ p_2,\ p_3,\ p_4;\ x_4p_1 - x_2p_3,\ x_1p_1 - x_2p_2 + x_3p_3 - x_4p_4,\ x_3p_2 - x_1p_4,\ x_3p_1 - x_2p_4, \\ x_1p_1 + 3(x_2p_2 + x_3p_3) + x_4p_4,\ x_3 . U - (x_1x_2 + x_3x_4)p_4,\ x_2 . U - (x_1x_2 + x_3x_4)p_1 . \end{array}}$$

This is a subgroup of the one last found, and is imprimitive for the same reason that that one is.

CHAPTER IV.

A Surface of the Second Degree and Two Coinciding Generators Invariant in R_3.

§1.

The projective group written in xyz has the form

$$r + xq,\ yq + zr,\ (xz - y)p + yzq + z^2r,\ p + zq,$$

and leaves the surface $y - zx = 0$

and two of its generators which coincide invariant. When written linear and homogeneous in $x_1x_2x_3x_4$ our group has the form

$$x_4p_1 - x_2p_3,\ x_1p_1 - x_2p_2 + x_3p_3 - x_4p_4,\ x_3p_2 - x_1p_4,\ x_3p_1 - x_2p_4,$$

to which may be added

$$U \equiv \sum_{1}^{4}{}^{i}\, x_ip_i .$$

§2.

Thus the transformations *IO* of the groups we seek in this chapter have the forms

$$S_1 \equiv x_4p_1 - x_2p_3,\ S_2 \equiv x_1p_1 - x_2p_2 + x_3p_3 - x_2p_4,\ S_3 \equiv x_3p_2 - x_1p_4,\ S_4 \equiv x_3p_1 - x_2p_4,$$

to which may be added

$$U \equiv \sum_{1}^{4}{}^{i}\, x_ip_i .$$

As to the transformations of sO which may occur in our group, we find immediately as in Chapter III, §2, that when s is the maximum order,

$$s < 3.$$

Let us see what transformations IIO can occur.

The calculations here are similar to those in the above-mentioned chapter and §.

Suppose

$$\sum_1^4 {}_i\, \xi_i^{(2)} p_i$$

is a transformation IIO which occurs, and let us suppose for the present that $\xi_2^{(2)} \neq 0$. Then we see a transformation IIO of form

$$Xf \equiv x_2^2 p_2 + \xi_3^{(2)}(x_2 x_3) p_3 + \xi_1^{(2)}(x_1 x_3 x_4) p_1 + \xi_4^{(2)}(x_2 x_4) p_4$$

must occur.

Combine Xf with S_2 and we find

$$Y \equiv x_2^2 p_2 + \left\{ x_2 \frac{\partial \xi_3}{\partial x_2} - x_3 \frac{\partial \xi_3}{\partial x_3} + \xi_3 \right\} p_3 + \left\{ x_2 \frac{\partial \xi_4}{\partial x_2} + x_4 \frac{\partial \xi_4}{\partial x_4} - \xi_4 \right\} p_4$$
$$- \left\{ x_3 \frac{\partial \xi_1}{\partial x_3} + x_1 \frac{\partial \xi_1}{\partial x_1} - x_4 \frac{\partial \xi_1}{\partial x_4} - \xi_1 \right\} p_1$$

must occur.

If Yf is not the same transformation as Xf, we see by subtraction that a transformation of the form

$$\eta_1^{(2)} p_1 + \eta_3^{(2)} p_3 + \eta_4^{(2)} p_4$$

must occur, where $\eta_3^{(2)}$ is $\neq 0$. Hence we see that

$$Zf \equiv x_2^2 p_3 + \eta_1^{(2)}(x_1 x_4) p_1 + \eta_4^{(2)}(x_2 x_4) p_4$$

must belong to our group. Combine Zf and S_2, so

$$-3x_2^2 p_3 - \left\{ x_4 \frac{\partial \eta_1}{\partial x_4} - x_1 \frac{\partial \eta_1}{\partial x_1} + \eta_1 \right\} p_1 - \left\{ x_4 \frac{\partial \eta_4}{\partial x_4} + x_2 \frac{\partial \eta_4}{\partial x_2} - \eta_4 \right\} p_4$$

occurs. Multiply Zf by 3, and add to the last transformation; thus we get

$$2\eta_1 - x_4 \frac{\partial \eta_1}{\partial x_4} + x_1 \frac{\partial \eta_1}{\partial x_1} = 0, \quad 4\eta_4 - x_4 \frac{\partial \eta_4}{\partial x_4} - x_2 \frac{\partial \eta_4}{\partial x_2} = 0.$$

From the last equation $$\eta_4 \equiv x_2^4 \bar{\eta}_4\left(\frac{x_4}{x_2}\right).$$

But since η_4 is only of the second degree, we have

$$\bar{\eta}_4 \equiv 0.$$

We now easily find by combining Zf with the p_k that also

$$\eta_1 \equiv 0.$$

Hence $$Zf \equiv x_2^2 p_3,$$

and we see at a glance that this transformation cannot belong to our group.

Thus Yf must be the same transformation as Xf, *i. e.*

$$x_2 \frac{\partial \xi_3}{\partial x_2} - x_3 \frac{\partial \xi_3}{\partial x_3} = 0, \quad x_1 \frac{\partial \xi_1}{\partial x_1} + x_3 \frac{\partial \xi_1}{\partial x_3} - x_4 \frac{\partial \xi_1}{\partial x_4} = 0, \quad x_2 \frac{\partial \xi_4}{\partial x_2} + x_4 \frac{\partial \xi_4}{\partial x_4} = 2\xi_4.$$

Hence $$\xi_3 = a x_2 x_3, \quad \xi_1 = b x_1 x_4 + c x_3 x_4, \quad \xi_4 = d x_2 x_4.$$

Thus $$Xf \equiv x_2^2 p_2 + a x_2 x_3 p_3 + (b x_1 x_4 + c x_3 x_4) p_1 + d x_2 x_4 p_4.$$

Combine Xf with the p_k and we find at once

$$a = b = c = d = 0.$$

Since $x_2^2 p_2$ cannot occur, we see that no transformation *IIO* can occur at all.

Thus in the case where $\xi_2^{(2)}$, in the transformation *IIO*,

$$\overset{4}{\underset{1}{\sum_i}} \xi_i^{(2)} p_i,$$

is $\neq 0$, no transformation *IIO* can occur in our group.

If $\xi_2^{(2)} = 0$, we have a transformation of the form

$$\eta_1^{(2)} p_1 + \eta_3^{(2)} p_3 + \eta_4^{(2)} p_4,$$

which, as we saw above, cannot occur.

Hence no transformations of an order higher than the first can occur in our groups.

We shall, as usual, distinguish the two cases where U occurs free and where U does not occur free.

§3.

Suppose U occurs free.

We normalize as usual with this transformation and find without any difficulty the group:

$$\boxed{\begin{array}{c} p_1 p_2 p_3 p_4;\ x_4p_1 - x_2p_3,\ x_1p_1 - x_2p_2 + x_3p_3 - x_4p_4,\ x_3p_2 - x_1p_4, \\ x_3p_1 - x_2p_4,\ x_1p_1 + x_2p_2 + x_3p_3 + x_4p_4. \end{array}}$$

This group is evidently imprimitive.

§4.

Suppose U does not occur free.

We easily see that in this case U can occur additively only in the form

$$S_4 + \alpha . U \qquad (\alpha \text{ const.}).$$

Thus our transformations IO are

$$S_1 \equiv x_4p_1 - x_2p_3,\quad S_2 \equiv x_1p_1 - x_2p_2 + x_3p_3 - x_4p_4,\quad S_3 \equiv x_3p_2 - x_1p_4,$$

$$T \equiv S_4 + \alpha . U \equiv x_3p_1 - x_2p_4 + \alpha . \sum_{1}^{4}{}_{i}\, x_ip_i .$$

Let us normalize with S_2. We easily find that we may without loss of generality choose the transformations zero O, so that

$$\left\{\begin{array}{rl} (p_1S_2) = & p_1, \\ (p_2S_2) = & -p_2, \\ (p_3S_2) = & p_3, \\ (p_4S_2) = & -p_4. \end{array}\right.$$

Now we wish to see how the p_k are connected with the S_i and with T. We have

$$(p_1S_1) \equiv \sum_{1}^{3}{}_{i}\, m_iS_i + mT.$$

Form Jacobi's Identity with p_1, S_1 and S_2, thus

$$((p_1S_1)\,S_2) - 3\,(p_1S_1) = 0,$$

i. e.

$$(p_1S_1) = 0.$$

Thus p_1 and S_1 are connected by a normal relation; in the same manner we find that the other $(p_k S_i)$ and $(p_k T)$ are normal.

It remains to see how the transformations zero O are connected among each other.

In the usual manner, by forming Jacobi's Identity with p_i, p_k and S_2, we find

$$\left\{\begin{array}{l}(p_1p_2)=b_2S_2+bT,\\(p_1p_3)=b_1S_1,\\(p_1p_4)=a_2S_2+aT,\\(p_2p_3)=\beta_2S_2+\beta T,\\(p_2p_4)=\gamma_3S_3,\\(p_3p_4)=c_2S_2+cT.\end{array}\right.$$

Now form Jacobi's Identity with p_1, p_2 and S_1, thus

$$((p_1p_2)\,S_1)+(p_1p_3)=0,$$

or

$$-2b_2S_1+b_1S_1=0,\quad b_1=2b_2.$$

Proceed in the same manner with p_i, p_k, S_j; p_i, p_k, T and p_i, p_k, p_j; then we find that our above constants must satisfy the relations

$$b_1=\gamma_3=b_2=c_2=a_2=\beta_2=a=b=c=0,$$
$$a.\beta=0.$$

Hence

$$(p_1p_2)=(p_1p_3)=(p_1p_4)=(p_2p_4)=(p_3p_4)=0;$$
$$(p_2p_3)=\beta\,.\,T.$$

These relations show that the transformations p_1, p_4 form an invariant G_2 in the groups we here seek. Hence are these groups *imprimitive.**

If $\beta \not= 0$, it is easy to see that we find the following very peculiar imprimitive group:

$$\boxed{\begin{array}{c}p_1,\; p_2,\; p_3+x_2x_3p_1-\dfrac{x_2^2}{2}p_4,\; p_4x_3p_1-x_2p_4,\; \left(x_4-\dfrac{x_2^2x_3}{2}\right)p_1-x_2p_3\\ +\dfrac{x_2^3}{6}p_4,\; x_1p_1-x_2p_2+x_3p_3-x_4p_4,\; x_3p_2-x_1p_4+\dfrac{x_3^3}{3}p_1.\end{array}}$$

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* Compare p. 329, note.

If, on the contrary, $\beta = 0$, our transformations are connected by normal relations, and we evidently get the imprimitive group:

$$\boxed{\begin{gathered} p_1, p_2, p_3, p_4;\ x_4p_1 - x_2p_3,\ x_1p_1 - x_2p_2 + x_3p_3 - x_4p_4,\ x_3p_2 - x_1p_4, \\ x_3p_1 - x_2p_4 + \alpha(x_1p_1 + x_2p_2 + x_3p_3 + x_4p_4). \end{gathered}}$$

CHAPTER V.

A Surface of the Second Degree and Two Different Generators of one Family Invariant in R_3.

§1.

The projective group written in the variables xyz has the form

$$r + xq,\ xp + yq,\ (xz - y)p + yzq + z^2r,\ yq + zr,$$

and leaves the surface of the second degree

$$y - zx = 0$$

and two different generators of one family on this surface invariant.

When written linear and homogeneous in $x_1x_2x_3x_4$, the group has the form

$$x_4p_1 - x_2p_3,\ x_1p_1 - x_2p_2 + x_3p_3 - x_4p_4;\ x_3p_2 - x_1p_4;\ x_1p_1 - x_2p_2 - x_3p_3 - x_4p_4,$$

to which may be added

$$U \equiv \sum_1^4{}_i\, x_ip_i.$$

§2.

Thus the transformations *IO* of the groups we seek now have the forms

$$\begin{gathered} x_4p_1 - x_3p_3 \equiv S_1,\ x_1p_1 - x_2p_2 + x_3p_3 - x_4p_4 \equiv S_2,\ x_3p_2 - x_1p_4 \equiv S_3, \\ x_1p_1 - x_2p_2 - x_3p_3 + x_4p_4 \equiv S_4, \end{gathered}$$

to which may be added
$$U \equiv \sum_{1}^{4}{}_i\, x_i p_i .$$

What transformations *IIO* can occur? If any transformation *IIO* occurs, it must have the form

$$\begin{aligned} &\xi_1^{(2)}(x_1x_4)p_1 + \xi_2^{(2)}(x_2x_3)p_2 + \xi_3^{(2)}(x_2x_3)p_3 + \xi_4^{(2)}(x_1x_4)p_4 \\ &\equiv (ax_1^2 + bx_1x_4 + cx_4^2)p_1 + (\ )p_2 + (\ )p_3 + (\ )p_4 . \end{aligned}$$

By combining this transformation with the p_k we easily see that all the constants ab . . . . are zero.

Thus, no transformation of an order higher than the first can occur in our groups.

We shall distinguish as usual the cases where U occurs free and where U does not occur free.

§3.

U occurs free.

Here we normalize with the transformation U as usual and find without difficulty the group:

$$\boxed{\begin{array}{c} p_1,\ p_2,\ p_3,\ p_4;\ x_4p_1 - x_2p_3,\ x_1p_1 - x_2p_2 + x_3p_3 - x_4p_4,\ x_3p_2 - x_1p_4, \\ x_1p_1 - x_2p_2 - x_3p_3 + x_4p_4,\ x_1p_1 + x_2p_2 + x_3p_3 + x_4p_4. \end{array}}$$

We see at once that this group is imprimitive since the family of $\infty^2 M_2$,

$$\begin{cases} x_2 = \text{const.} \\ x_3 = \text{const.} \end{cases}$$

as well as that of
$$\begin{cases} x_1 = \text{const.} \\ x_4 = \text{const.} \end{cases}$$
are invariant.

§4.

U does not occur free.

It is easy to see that U can occur additively only in the form

$$S_4 + \alpha . U, \qquad (\alpha = \text{const.}).$$

Our transformations IO are

$$S_1 \equiv x_4p_1 - x_2p_3, \quad S_2 \equiv x_1p_1 - x_2p_2 + x_3p_3 - x_4p_4, \quad S_3 \equiv x_3p_2 - x_1p_4,$$
$$S_4 + \alpha . U \equiv x_1p_1 - x_2p_2 + x_3p_3 + x_4p_4 + \alpha . \sum_1^4 {}^i x_ip_i \equiv T.$$

Let us normalize with S_2. We find as usual without trouble.

$$\begin{cases} (p_1S_2) = p_1, \\ (p_2S_2) = -p_2, \\ (p_3S_2) = p_3, \\ (p_4S_2) = -p_4. \end{cases}$$

We find in the ordinary manner that all the other relations (p_kS_i), (p_kT) are normal.

We wish to see how the transformations of the zero O are connected. We find in the usual manner by forming Jacobi's Identity with p_i, p_k, and S_2,

$$\begin{cases} (p_1p_2) = b_2S_2 + bT, \\ (p_1p_3) = b_1S_1, \\ (p_1p_4) = a_2S_2 + aT, \\ (p_2p_3) = \beta_2S_2 + \beta T, \\ (p_2p_4) = \gamma_3S_3, \\ (p_3p_4) = c_2S_2 + cT. \end{cases}$$

By forming further Jacobi's Identity with p_i, p_k, S_j; p_i, p_k, T and p_i, p_k, p_j, we find for our above constants the relations

$$a = a_2 = \beta = \beta_2 = 0, \quad c = -b, \quad b_1 = \gamma_3 = 2b_2 = 2c_2, \quad b_2 - b + \gamma_3 = 0,$$
$$\alpha . b = \alpha . b_2 = 0.$$

If $\alpha \neq 0$, we see that all constants are zero, and our transformations are all connected by normal relations. This gives the group:

$$\boxed{\begin{gathered} p_1, p_2, p_3, p_4; \; x_4p_1 - x_2p_3; \; x_1p_1 - x_2p_2 + x_3p_3 - x_4p_4; \; x_3p_2 - x_1p_4; \\ x_1p_1 - x_2p_2 - x_3p_3 + x_4p_4 + \alpha (x_1p_1 + x_2p_2 + x_3p_3 + x_4p_4). \end{gathered}}$$

If, on the contrary, $\alpha = 0$, we have

$$\begin{aligned}(p_1p_2) &= b_2(S_2 + 3S_4),\\ (p_1p_3) &= 2b_2S,\\ (p_1p_4) &= 0;\\ (p_2p_3) &= 0,\\ (p_2p_4) &= 2b_2S_3,\\ (p_3p_4) &= b_2(S_2 - 3S_4).\end{aligned}$$

Here we may suppose $b_2 \neq 0$, otherwise we would get the above group again. We may arrange so that $b_2 = 1$ if we put $b_2 . p_2$ for a new p_2, and $b_2 . p_3$ for a new p_3. This gives the group:

$$\boxed{\begin{array}{l} p_1,\ p_2 + 2x_1^2p_1 - (4x_1x_2 - 2x_3x_4)p_2 - 2x_1x_3p_3 + 2x_1x_4p_4,\\ p_4,\ p_3 + 2x_1x_4p_1 - 2x_2x_4p_2 - (4x_3x_4 + 2x_1x_2)p_3 + 2x_4^2p_4,\\ x_4p_1 - x_2p_3,\ x_1p_1 - x_2p_2 + x_3p_3 - x_4p_4,\ x_3p_2 - x_1p_4,\\ x_1p_1 - x_2p_2 - x_3p_3 + x_4p_4. \end{array}}$$

Both the groups in this § are imprimitive, since the family of ∞^2 manifoldnesses

$$\begin{cases} x_1 = \text{const.}\\ x_4 = \text{const.} \end{cases}$$

is invariant under both.

CHAPTER VI.

A Surface of the Second Degree and all Generators of One Family Invariant in R_3.

§1.

The projective group written in the variables xyz has the form

$$r + xq,\ yq + zr,\ (xz - y)p + yzq + z^2r,$$

and leaves the surface of the second degree

$$y - zx = 0$$

and each generator, of one family, of this surface invariant.

Written linear and homogeneous in the variables $x_1x_2x_3x_4$ our group has the form

$$x_4p_1 - x_2p_3, \quad x_1p_1 - x_2p_2 + x_3p_3 - x_4p_4, \quad x_3p_2 - x_1p_4,$$

to which may be added

$$U \equiv \sum_1^4{}^i x_ip_i.$$

§2.

Thus the transformations IO of the groups we now seek have the forms

$$S_1 \equiv x_4p_1 - x_2p_3, \ S_2 \equiv x_1p_1 - x_2p_2 + x_3p_3 - x_4p_4, \ S_3 \equiv x_3p_2 - x_1p_4, \ U \equiv \sum_1^4{}^i x_ip_i,$$

where U may or may not occur. For the initial terms of these transformations the relations hold,

$$(S_1S_2) = 2S_1, \ (S_1S_3) = S_2, \ (S_2S_3) = 2S_3, \ (S_iU) = 0.$$

Thus we see that if U does not occur free, it cannot occur additively.

Let us see what transformations IIO can occur.

We easily see that the general transformation IIO has the form

$$\begin{aligned} Xf &\equiv \xi_1^{(2)}(x_1x_4)p_1 + (\xi_2^{(2)}(x_2x_3)p_2 + \xi_3^{(2)}(x_2x_3)p_3 + \xi_4^{(2)}(x_1x_4)p_4 \\ &\equiv (a_1x_1^2 + bx_1x_4 + cx_4^2)p_1 + (\ )p_2 + (\ )p_3 + (\ )p_4, \end{aligned}$$

where a, b, c . . . . are certain constants. If now we combine Xf with p_1, p_2, p_3, p_4 and express the results linearly in the transformation IO, we find

$$a = b = c = \ldots . = 0.$$

Thus no transformation IIO can occur, and so none of an order higher than the first can occur.

As usual, we have two cases according as U occurs or not.

§3.

Suppose U occurs.

Here we normalize with U as in the preceding chapters, and without any difficulty find the group:

$$\boxed{\begin{gathered} p_1, p_2, p_3, p_4; \ x_4p_1 - x_2p_3, \ x_1p_1 - x_2p_2 + x_3p_3 - x_4p_4, \\ x_3p_2 - x_1p_4, \ x_1p_1 + x_2p_2 + x_3p_3 + x_4p_4. \end{gathered}}$$

This group is imprimitive, since the family of $\infty^2 M_2$,

$$\begin{cases} x_1 = \text{const.} \\ x_4 = \text{const.} \end{cases}$$

as well as that of

$$\begin{cases} x_2 = \text{const.} \\ x_3 = \text{const.} \end{cases}$$

are invariant.

§4.

Suppose U does not occur.

Let us normalize here with S_2. Our transformations are

$$p_1, p_2, p_3, p_4; \; x_4p_1 - x_2p_3 \equiv S_1, \; x_1p_1 - x_2p_2 + x_3p_3 - x_4p_4 \equiv S_2, \; x_3p_2 - x_1p_4 \equiv S_3.$$

We easily see that, just as in §3, Chap. I, we may choose, without loss of generality, the transformations zero O, so that

$$\begin{cases} (p_1S_2) = p_1, \\ (p_2S_2) = -p_2, \\ (p_3S_2) = p_3, \\ (p_4S_2) = -p_4. \end{cases}$$

Now let us see how the transformations zero O are connected with the other transformations IO. We have for example

$$(p_1S_1) = \sum_1^3{}^i \, a_iS_i, \qquad (a_i \text{ const.}).$$

Form Jacobi's Identity with p_1S_1 and S_2, thus

$$((p_1S_1)\,S_2) - 3\,(p_1S_1) = 0,$$

or

$$(p_1S_1) = 0.$$

Thus (p_1S_1) is normal; same way we find that the other (p_kS_i) are also normal.

It remains to see how the transformations zero O are connected among each other. We have

$$(p_1p_2) = \sum_1^4{}^i \, a_ip_i + \sum_1^3{}^j \, b_jS_j.$$

Form Jacobi's Identity with p_1, p_2 and S_2, thus

$$((p_1p_2)\,S_2) \equiv 0,$$

or

$$a_i = b_1 = b_3 = 0$$

and

$$(p_1p_2) = b_2S_2.$$

Same way
$$\begin{cases} (p_1p_3) = a_1S_1, \\ (p_1p_4) = d_2S_2, \\ (p_2p_3) = c_2S_2, \\ (p_2p_4) = e_3S_3, \\ (p_3p_4) = f_2S_2. \end{cases}$$

Now form Jacobi's Identity with p_1, p_2 and S_3, thus
$$((p_1p_2)\,S_3) - (p_2p_4) = 0,$$
or
$$2b_2S_3 - e_3S_3 = 0,\ \ 2b_2 = e_3.$$
Same way
$$((p_1p_2)\,S_1) + (p_1p_3) = 0,$$
or
$$-2b_2S_1 + a_1S_1 = 0,\ \ 2b_2 = a_1.$$
Further
$$-(p_3\,(p_1p_2)) - (p_1\,(p_2p_3)) + (p_2\,(p_1p_3)) \equiv 0,$$
or
$$-b_2p_3 - c_2p_1 - a_1p_3 = 0,\ \ b_2 = -a_1,\ \ c_2 = 0.$$
Thus
$$b_2 = a_1 = e_3 = c_2 = 0,$$
and we easily find in the same way that
$$d_2 = f_2 = 0.$$

Thus all of our transformations are connected by normal relations, and we evidently get the group:

$$\boxed{p_1, p_2, p_3, p_4;\ x_4p_1 - x_2p_3,\ x_1p_1 - x_2p_2 + x_3p_3 - x_4p_4,\ x_3p_2 - x_1p_4.}$$

This is a subgroup of the one found in the last §, and is imprimitive for the same reason that that one is.

The necessary limits of this article do not permit me to give at present the calculations in the four cases which remain of this problem. Suffice it to say that I have finished these somewhat long and difficult calculations, and have shown that no primitive groups, except those already known, exist in the space of four dimensions. I hope to publish the calculations themselves at a later date, thereby furnishing a solution of a considerable part of the important and extensive problem of *finding* ALL *groups, primitive and imprimitive, in* R_4.

Synopsis of all Primitive Groups in R_4.

We shall designate those groups, which Lie has determined, as he has done, and refer to his works for an explanation of the names, when such is needed.

(A). *Groups which transform the directions through a point, of general position in R_4, which we hold, in the most general manner possible.*

1). $$\boxed{p_k;\ x_i p_k;\ x_1 U;\ x_2 . U;\ x_3 . U,\ x_4 . U;\ i,\ k = 1 \ldots . 4.}$$

Here as usual U denotes $\sum_1^4{}^i x_i p_i$. This is the *general projective group* in R_4.

2). $$\boxed{p_k,\ x_i p_k;\ i,\ k = 1 \ldots . 4.}$$

This is the *general linear group.*

3). $$\boxed{p_k,\ x_i p_k,\ x_i p_k - x_k p_k;\ i \neq k;\ i,\ k = 1 \ldots . 4.}$$

This is the *special linear group.*

(B). *Groups which leave a Surface of the Second Degree in R_3 invariant.*

1). $$\boxed{p_i,\ x_i p_k - x_k p_i;\ i,\ k = 1 \ldots . 4.}$$

This is the group of *Euclidean motions* (*Euklidische Bewegungen*).

2). $$\boxed{p_i,\ x_i p_k - x_k p_i,\ \sum_1^4{}^j x_j p_j;\quad i,\ k = 1 \ldots . 4.}$$

This is the group of *Euclidean motions and similar transformations* (*Euklidische Bewegungen und Ähnlichkeitstoff*).

3). $$\boxed{p_i - x_i . U,\ x_i p_k - x_k p_i;\quad i,\ k = 1 \ldots . 4.}$$

This group leaves a surface of second degree

$$\sum_1^4 {}_i\, x_i^2 = 1$$

in R_4 invariant.

4).

$$\boxed{p_i,\ x_i p_k - x_k p_i,\ U,\ 2x_i U - p_i . \sum_1^4 {}_i\, x_i^2; \quad i, k = 1 \ldots . 4.}$$

This is the group of *conform transformations.*

(C). *Groups which leave a twisted curve IIIO in R_3 invariant.*

1).

$$\boxed{\begin{gathered} p_k,\ x_1p_1 - x_2p_2 + 3\,(x_4p_4 - x_3p_3),\ x_1p_1 + x_2p_2 + x_3p_3 + x_4p_4, \\ x_4p_1 + 2x_1p_2 + 3x_2p_3,\ 2x_2p_1 + x_3p_2 + 3x_1p_4. \end{gathered}}$$

2).

$$\boxed{\begin{gathered} p_k,\ x_4p_1 + 2x_1p_2 + 3x_2p_3,\ 2x_2p_1 + x_3p_2 + 3x_1p_4, \\ x_1p_1 - x_2p_2 + 3\,(x_4p_4 - x_3p_3). \end{gathered}}$$

(D). *Groups which leave a linear complex in R_3 invariant.*

1).

$$\boxed{\begin{gathered} p_k,\ x_4p_1 - x_2p_3,\ x_4p_2 + x_1p_3,\ x_4p_3,\ x_1p_1 - x_2p_2,\ x_2p_1,\ x_1p_2,\ x_3p_1 + x_2p_4, \\ x_3p_2 - x_1p_4,\ x_3p_4,\ x_3p_3 - x_4p_4,\ x_1p_1 + x_2p_2 + x_3p_3 + x_4p_4. \end{gathered}}$$

2).

$$\boxed{\begin{gathered} p_k,\ x_4p_1 - x_2p_3,\ x_4p_2 + x_1p_3,\ x_4p_3,\ x_1p_2,\ x_1p_1 - x_2p_2,\ x_2p_1, \\ x_3p_1 + x_2p_4,\ x_3p_2 - x_1p_4,\ x_3p_4,\ x_1p_1 - x_2p_2 + x_3p_3 - x_4p_4. \end{gathered}}$$

These are all the primitive groups in space of four dimensions: only eleven in number.

Line Congruences.

By W. C. L. Gorton, *Fellow at the Johns Hopkins University.*

This subject, which has already been otherwise treated by Hamilton and Kummer (Crelle, Vol. LVII), I propose to treat by quaternions. I have obtained all the results given by Kummer in the above-mentioned paper, and have also been enabled by the method employed to carry out certain steps which he has only indicated. Since in this paper the straight line is determined as a function of two independent parameters, there will be a double infinity of the straight lines in space, and through each point in space will pass one or more rays. Every ray will be determined by a point through which it passes which we call the initial point, and by its direction. We will take a surface cutting all the rays as the locus of the initial points, and then through each point of the surface will pass one or more rays of the system. Let $\sigma = \phi(t, u)$ be the equation of this surface, and let the direction of the ray be given by $\tau = \frac{\psi(t, u)}{T\psi(t, u)}$, where ϕ and ψ are vector functions of t and u. The locus of the extremity of τ will then be a sphere of radius unity, and when we refer to the surface τ we will mean this sphere. The equation of any ray then is of the form

$$\rho = \sigma + x\tau.$$

§1.

The Foci and Focal Planes.

Let $\rho = \sigma + x\tau$ be the equation of any ray, then any consecutive ray will be of the form
$$\rho = \sigma + d\sigma + y(\tau + d\tau).$$
In general this ray will not cut the ray τ, which we will refer to as the primitive ray. In order that it may, we have the necessary and sufficient condition,
$$d\rho = d\sigma + xd\tau + dx\tau = 0,$$
which gives us
$$S\tau d\sigma d\tau = 0, \tag{1}$$

a quadratic in dt and du; which will therefore give us two values of the ratio of dt to du. This quadratic may not always have its roots real. They will be real or imaginary according to the law which unites the straight lines into one system of rays. Therefore we have two kinds of systems of rays, one in which every ray is cut by two consecutive rays, and the other in which a ray is generally cut by no consecutive ray. As a third kind of a system, we have that in which certain rays belong to the first and others to the second kind mentioned above. The points in which consecutive rays cut the primitive ray we will call the foci of this ray, and the planes of these rays and the primitive ray we will call the focal planes. To obtain the distances of the foci from the initial point of τ we have

$$d\sigma = \partial_t\sigma\, dt + \partial_u\sigma du$$

and

$$d\tau = \partial_t\tau\, dt + \partial_u\tau du,$$

$$d\rho = d\sigma + x d\tau + dx\tau = 0$$

$$= (\partial_t\sigma + x\partial_t\tau)\, dt + (\partial_u\sigma + x\partial_u\tau)\, du + dx\tau = 0,$$

which gives us

$$S\tau\,(\partial_t\sigma + x\partial_t\tau)(\partial_u\sigma + x\partial_u\tau) = 0. \qquad (2)$$

If we call the roots of (1) $\frac{dt_1}{du_1}$ and $\frac{dt_2}{du_2}$, then

$$d\tau_1 = \partial_t\tau\, dt_1 + \partial_u\tau du_1,$$

$$d\tau_2 = \partial_t\tau\, dt_2 + \partial_u\tau du_2$$

we will call the focal lines.

For the ray in question we will take

$$S\partial_u\tau\partial_t\tau = 0 \text{ and } T\partial_t\tau = T\partial_u\tau = 1.$$

Since we have two independent parameters, if $\partial_t\tau$ and $\partial_u\tau$ are not perpendicular, let us take t and u functions of two new parameters, say t' and u'.

Let

$$t = \eta\,(t', u') = a_1 + a_2t' + a_3u' + \ldots.,$$

$$u = \zeta\,(t', u') = b_1 + b_2t' + b_3u' + \ldots.$$

Then

$$dt = \eta'_{t'}dt' + \eta'_{u'}du'$$

and

$$du = \zeta'_{t'}dt' + \zeta'_{u'}du'$$

and

$$d\tau = (\eta'_{t'}\partial_t\tau + \zeta'_{t'}\partial_u\tau)\, dt' + (\eta'_{u'}\partial_t\tau + \zeta'_{u'}\partial_u\tau)\, du'.$$

We may assume any value of t' and u' for this ray, say 0, 0; for if they had the values t'_1 and u'_1 we could take $t' - t'_1$ and $u' - u'_1$ as new variables. Then for this ray we have

$$\eta'_{t'} = a_2,\ \eta'_{u'} = a_3,\ \zeta'_{t'} = b_2 \text{ and } \zeta'_{u'} = b_3,$$

since the derivatives of the terms containing t' and u' to more that the first degree would contain t' or u' and therefore vanish. We have then

$$d\tau = (a_2\partial_t\tau + b_2\partial_u\tau)\,dt' + (a_3\partial_t\tau + b_3\partial_u\tau)\,du'.$$

In any case $S\tau d\tau = 0$ since $T\tau = 1$, and therefore $\partial_t\tau$ and $\partial_u\tau$ are always perpendicular to τ. If $\partial_t\tau$ and $\partial_u\tau$ have not the same direction, since we have at our disposal four arbitrary constants, we can make

$$a_2\partial_t\tau + b_2\partial_u\tau \text{ and } a_3\partial_t\tau + b_3\partial_u\tau$$

have any directions perpendicular to τ and any lengths we please. It is evident that we can make $\partial_t\tau$ and $\partial_u\tau$ perpendicular for any ray since we have an infinity of constants at our disposal. We will so choose a_2, a_3, b_2 and b_3 that $\partial_t\tau$ and $\partial_u\tau$ will be the bisectors (of length unity) of the angles between the focal lines. We can evidently always make this choice, since if the roots of (1) are not real they are conjugate imaginaries, and hence the bisectors of the angles between the focal lines are always real. Thus we have now

$$S\partial_u\tau\partial_t\tau = 0, \quad T\partial_u\tau = T\partial_t\tau = 1$$

for the ray in question, and in every case, since the tensor of τ is constant and equal to unity, $S\tau d\tau = 0$. Let us choose also as the surface σ (the locus of initial points) the locus of the point midway between the foci on any ray. This point we will call the centre of the ray, and thus σ is the locus of centres. Let us now see what simplification these assumptions will introduce into our formulae. Since $\partial_t\tau$ and $\partial_u\tau$ are the bisectors of the angles between the focal planes, we must have the sum of the roots of (1) zero, therefore

$$S\tau\partial_t\sigma\partial_u\tau - S\tau\partial_t\tau\partial_u\sigma = 0.$$

Also, since we have taken the surface σ as the locus of centres, the sum of the roots of (2) is zero,

$$\therefore\ S\tau\partial_t\sigma\partial_u\tau + S\tau\partial_t\sigma\partial_u\sigma = 0,$$

which two equations give us

$$S\tau\partial_t\sigma\partial_u\tau = 0$$

and

$$S\tau\partial_t\tau\partial_u\sigma = 0.$$

Putting for τ, $\partial_t\tau\partial_u\tau$, we have

$$S\tau\partial_t\sigma\partial_u\tau = S\partial_t\tau\partial_u\tau\partial_t\sigma\partial_u\tau = -(\partial_u\tau)^2 S\partial_t\sigma\partial_t\tau = S\partial_t\sigma\partial_t\tau \text{ since } (\partial_u\tau)^2 = -1.$$

Also

$$S\tau\partial_t\tau\partial_u\sigma = -S\partial_t\tau\tau\partial_u\sigma = -S\partial_t\tau\partial_t\tau\partial_u\tau\partial_u\sigma = -S\partial_u\sigma\partial_u\tau,$$

$$\therefore\ S\partial_t\sigma\partial_t\tau = 0, \tag{3}$$

$$S\partial_u\sigma\partial_u\tau = 0. \tag{4}$$

§2.

The Vertices of the Ray and the Principal Planes.

We will now find the distance from the initial point to the foot of the common perpendicular to τ and any consecutive ray. The direction of the common perpendicular is $\tau d\tau$. We must have

$$x\tau = d\sigma + y(\tau + d\tau) + z\tau d\tau.$$

Operating by $S(\tau + d\tau)\tau d\tau$ and neglecting terms of the third order, we have

$$x(d\tau)^2 = -Sd\sigma d\tau,$$

$$\therefore\ x = -\frac{Sd\sigma d\tau}{(d\tau)^2}.$$

Since
$$d\sigma = \partial_t\sigma dt + \partial_u\sigma du$$
and
$$d\tau = \partial_t\tau dt + \partial_u\tau du,$$
we have
$$x = \frac{dtdu\,(S\partial_u\sigma\partial_t\tau + S\partial_t\sigma\partial_u\tau)}{dt^2 + du^2}. \tag{1}$$

For any value of the ratio of dt to du this expression gives the distance x of the centre of the ray from the foot of the common perpendicular to τ and the consecutive ray corresponding to this value of $\frac{dt}{du}$. By giving $\frac{dt}{du}$ all values from $-\infty$ to $+\infty$ we will get all the different values of x. For real values of dt and du the denominator of x cannot become zero, and therefore x cannot become infinite; hence it must vary between two limits, a maximum and minimum value. For a maximum or minimum value of x we must have $\frac{\delta x}{\delta \frac{dt}{du}} = 0$, and therefore we have for a maximum or minimum

$$dt^2 + du^2 - 2dt^2 = 0,$$

or

$$\left(\frac{dt}{du}\right)^2 = 1,$$

$$\left(\frac{dt}{du}\right) = \pm 1. \tag{2}$$

To obtain the two values of x corresponding to these values of the ratio of $\frac{dt}{du}$, since we have $\frac{\delta x}{\delta dt} = 0$ and $\frac{\delta x}{\delta du} = 0$, we must have

$$2xdt - (S\partial_t\tau\partial_u\sigma + S\partial_u\tau\partial_t\sigma)\,du = 0$$

and

$$2xdu - (S\partial_t\tau\partial_u\sigma + S\partial_u\tau\partial_t\sigma)\,dt = 0,$$

and therefore

$$\begin{vmatrix} 2x, & S\partial_t\tau\partial_u\sigma + S\partial_u\tau\partial_t\sigma \\ S\partial_t\tau\partial_u\sigma + S\partial_u\tau\partial_t\sigma, & 2x \end{vmatrix} = 0,$$

or

$$x^2 - \left(\frac{S\partial_t\tau\partial_u\sigma + S\partial_u\tau\partial_t\sigma}{2}\right)^2 = 0. \tag{3}$$

The points so determined we will call the vertices of the ray. By taking $\frac{dt}{du} = \pm 1$ we get

$$d\tau_1 = (\partial_t\tau + \partial_u\tau)\,dt$$

and

$$d\tau_2 = (\partial_t\tau - \partial_u\tau)\,dt.$$

These lines we will call the principal lines. The planes of τ and the principal lines we will call the principal planes.

From (2) and (3) we get the following theorems:

I. The principal lines bisect the angles between $\partial_t\tau$ and $\partial_u\tau$, and therefore the principal planes are at right angles.

II. The point midway between the vertices coincides with the point midway between the foci. This point we have called the centre of the ray.

Let us call the roots of (2), §1, ρ_1 and ρ_2, and the roots of (3), §2, r_1 and r_2.

$$\rho_1 = +\sqrt{S\partial_t\tau\partial_u\sigma S\partial_u\tau\partial_t\sigma},$$

$$\rho_2 = -\sqrt{S\partial_t\tau\partial_u\sigma S\partial_u\tau\partial_t\sigma},$$

$$r_1 = \frac{S\partial_t\tau\partial_u\sigma + S\partial_u\tau\partial_t\sigma}{2},$$

$$r_2 = -\frac{S\partial_t\tau\partial_u\sigma + S\partial_u\tau\partial_t\sigma}{2},$$

we have

$$\rho_1\rho_2 - r_1r_2 = \left(\frac{S\partial_t\sigma\partial_u\tau - S\partial_u\sigma\partial_t\tau}{2}\right)^2, \tag{4}$$

or since

$$\rho_1 = -\rho_2 \text{ and } r_1 = -r_2,$$

$$r_1^2 - \rho_1^2 = \left(\frac{S\partial_t\sigma\partial_u\tau - S\partial_u\sigma\partial_t\tau}{2}\right)^2.$$

We have thus the theorem that the distance of the foci from the centre is never greater than the distance of the vertices from the centre. We have for r, the distance from the initial point to the foot of the common perpendicular to τ and any consecutive ray,

$$r = -\frac{Sd\sigma d\tau}{(d\tau)^2} = \frac{dtdu\,(S\partial_u\tau\partial_t\sigma + S\partial_t\tau\partial_u\sigma)}{dt^2 + du^2}.$$

Let us call ω the angle which the common perpendicular makes with the first principal line. $d\tau_1 = (\partial_t\tau + \partial_u\tau)\,dt_1$.

$$\cos^2\omega = \frac{S^2 d\tau d\tau_1}{(d\tau)^2(d\tau_1)} = -\frac{(dt+du)^2}{2\,(d\tau)^2},$$

$$\sin^2\omega = \frac{S^2 d\tau d\tau_2}{(d\tau)^2(d\tau_2)^2} = -\frac{(dt-du)^2}{2\,(d\tau)^2},$$

$$r_1\cos^2\omega = -\frac{S\partial_t\tau\partial_u\sigma + S\partial_u\tau\partial_t\sigma}{2}\,\frac{(dt+du)^2}{2\,(d\tau)^2},$$

$$r_2\sin^2\omega = \frac{S\partial_t\tau\partial_u\sigma + S\partial_u\tau\partial_t\sigma}{2}\,\frac{(dt-du)^2}{2\,(d\tau)^2},$$

$$\therefore\; r_1\cos^2\omega + r_2\sin^2\omega = -\frac{dudt\,(S\partial_t\tau\partial_u\sigma + S\partial_u\tau\partial_t\sigma}{(d\tau)^2}$$
$$= r.$$

Therefore we have the neat formula

$$r = r_1\cos^2\omega + r_2\sin^2\omega.$$

All these results have been obtained by both Hamilton and Kummer.

For the foci we have

$$d\tau_1 \parallel \partial_t\tau\sqrt{S\partial_u\sigma\partial_t\tau} + \partial_u\tau\sqrt{S\partial_u\tau\partial_t\sigma},$$
$$d\tau_2 \parallel \partial_t\tau\sqrt{S\partial_u\sigma\partial_t\tau} - \partial_u\tau\sqrt{S\partial_u\tau\partial_t\sigma};$$

for the principal lines

$$d\tau_1 = (\partial_t\tau + \partial_u\tau)\,dt_1,$$
$$d\tau_2 = (\partial_t\tau - \partial_u\tau)\,dt_2.$$

We have then for the equations of the focal planes,

$$S(\rho-\sigma)\,\tau\,(\partial_t\tau\sqrt{S\partial_u\sigma\partial_t\tau} + \partial_u\tau\sqrt{S\partial_u\tau\partial_t\sigma}) = 0$$

and

$$S(\rho-\sigma)\,\tau\,(\partial_t\tau\sqrt{S\partial_u\sigma\partial_t\tau} - \partial_u\tau\sqrt{S\partial_u\tau\partial_t\sigma}) = 0.$$

Putting $\tau = \partial_u\tau\partial_t\tau$, we have for these equations

$$S(\rho-\sigma)\{\partial_t\tau\sqrt{S\partial_u\tau\partial_t\sigma} - \partial_u\tau\sqrt{S\partial_u\sigma\partial_t\tau}\} = 0, \tag{6}$$

and

$$S(\rho-\sigma)\{\partial_t\tau\sqrt{S\partial_u\tau\partial_t\sigma} + \partial\,\tau\sqrt{S\partial_u\sigma\partial_t\tau}\} = 0. \tag{7}$$

For the principal planes we have

$$S(\rho-\sigma)(\partial_t\tau-\partial_u\tau)=0, \tag{8}$$

$$S(\rho-\sigma)(\partial_t\tau+\partial_u\tau)=0. \tag{9}$$

These equations have not been given by Kummer.

§3.

The Surfaces connected with the System of Rays.

The foot-points on every ray of the system, namely, the two foci, the vertices and the common middle point, determine five surfaces as their loci. The two surfaces on which the vertices lie present themselves analytically as but one surface, in so far as they are both represented by one equation, but they may be looked upon as two surfaces or the two parts of one surface. Let us call the two surfaces F_1 and F_2. These two surfaces so divide space that between them lie the feet of the common perpendiculars to any ray and all its consecutive rays, but outside of them none.

When we have taken the surface σ as the locus of centres and not making the assumption $T\partial_t\tau=T\partial\,\tau=1$, we have the equations of the principal surfaces,

$$\rho=\sigma+\tau\sqrt{\frac{(S\partial_t\tau\partial_u\sigma-S\partial_u\tau\partial_t\sigma)^2-4S(V\partial_u\tau\partial_t\tau)\,\partial_t\sigma\partial_u\sigma}{4S(V\partial_u\tau\partial_t\tau)\,\partial_t\tau\partial_u\tau}}, \tag{1}$$

$$\rho=\sigma-\tau\sqrt{\frac{(S\partial_t\tau\partial_u\sigma-S\partial_u\tau\partial_t\sigma)^2-4S(V\partial_u\tau\partial_t\tau)\,\partial_t\sigma\partial_u\sigma}{4S(V\partial_u\tau\partial_t\tau)\,\partial_t\tau\partial_u\tau}}. \tag{2}$$

The two surfaces on which the foci of every ray lie, which we call the focal surfaces and which we will designate by ϕ_1 and ϕ_2, exist as real surfaces only when the roots of (1), §1 are real. Their equations are

$$\rho=\sigma+\tau\sqrt{\frac{S\tau\partial_u\sigma\partial_t\sigma}{S\tau\partial_t\tau\partial_u\tau}} \qquad (3)$$

$$\rho=\sigma-\tau\sqrt{\frac{S\tau\partial_u\sigma\partial_t\sigma}{S\tau\partial_t\tau\partial_u\tau}}.$$

We may now find what relation the rays have to this surface. For the tangent plane to ϕ_1 at the first focus whose distance from the centre is ρ_1,

$$d\rho=\left(\partial_t\sigma+\rho_1\partial_t\tau+\tau\frac{\partial\rho_1}{\partial t}\right)dt+\left(\partial_u\sigma+\rho_1\partial_u\tau+\tau\frac{\partial\rho_1}{\partial u}\right)du,$$

$$\begin{aligned}S\tau\partial_t\rho\partial_u\rho&=S\tau\left(\partial_t\sigma+\rho_1\partial_t\tau+\tau\frac{\partial\rho_1}{\partial t}\right)\left(\partial_u\sigma+\rho_1\partial_u\tau+\tau\frac{\partial\rho_1}{\partial u}\right)\\&=S\tau\,(\partial_t\sigma+\rho_1\partial_t\tau)(\partial_u\sigma+\rho_1\partial_u\tau);\end{aligned}$$

but since ρ_1 is a root of

$$S\tau\,(\partial_t\sigma + x\partial_t\tau)(\partial_u\sigma + x\partial_u\tau) = 0$$

we have

$$S\tau\partial_t\rho\partial_u\rho = 0; \tag{5}$$

∴ every ray touches each focal surface, and the system of rays may be therefore looked upon as a system of common tangents to two surfaces, or a system of double tangents to a single surface. If the two surfaces ϕ_1 and ϕ_2 intersect, then every straight line tangent to the curve of intersection will be a ray of the system since it touches each surface once, namely, in the point at which it is tangent to the curve. The middle points lie on a surface whose equation is

$$\rho = \sigma + \frac{\rho_1 + \rho_2}{2}\tau,$$

or

$$\rho = \sigma + \frac{S\tau\,(\partial_t\tau\partial_u\sigma + \partial_t\sigma\partial_u\tau)}{2S\tau\partial_u\tau\partial_t\tau}\tau. \tag{6}$$

This surface is always real.

All these surfaces may in special cases degenerate into lines or even into points. As a special case of a system of rays we may have all the straight lines through a point, in which case the focal surface is nothing but a point. Also the rays may be normals to a surface, in which case, as we shall see afterwards, the focal and principal surfaces coincide.

§4.

The Measure of Density.

Since $T\tau = 1$, the locus of the extremity of τ is a sphere of radius unity. Then for each ray of the system we will have a corresponding point on the sphere, and for each continuous succession of rays a continuous curve on the sphere. If at any point of a ray we pass a plane perpendicular to the ray and take a closed curve, in this plane, then to the group of rays which pass through this curve we will have a corresponding closed curve on the sphere. If we take the closed curve so that its points are at an infinitesimal distance from the ray, then we will have a corresponding infinitesimal curve on the sphere. The ratio of the area of the curve on the sphere to that of the curve in the plane (which in the case when the system of rays is normal to a surface, and the perpendicular plane a tangent plane, has been called by Gauss the measure of absolute curvature) we will call the measure of density. If at any point of a ray we pass a plane through it perpendicular to the ray, which will cut the bundle of consecutive rays in a curve whose area we will call f, and the area of the corresponding curve on the sphere we call ϕ, then the ratio $\frac{\phi}{f}$ is the measure of density of the system of rays at this point. We have

$$\phi = \frac{1}{2}\int TV.d\tau\,(d\tau + \delta d\tau) = \frac{1}{2}\int TV.d\tau\delta d\tau,$$

where $$d\tau = \partial_t\tau dt + \partial_u\tau du$$

and $$\delta d\tau = \partial_t\tau\delta dt + \partial_u\tau\delta du,$$

$$\therefore\ \phi = \frac{1}{2}\int(dt\delta du - du\delta dt). \tag{1}$$

For the perpendicular $d\sigma + x(\tau + d\tau)$ on τ we have

$$\begin{aligned} &= \tau V\tau\{d\sigma + x\tau + xd\tau\} \\ &= -xd\tau + \tau V\tau d\sigma. \end{aligned}$$

Then for the equation of the section of the bundle of rays by a plane perpendicular to τ and distant x from the centre we have

$$\rho = xd\tau + \tau Vd\sigma\tau, \tag{2}$$

$$\begin{aligned} \therefore\ f &= \frac{1}{2}\int TV.(xd\tau + \tau Vd\sigma\tau)(x\delta d\tau + \tau V\delta d\sigma\tau)\,xd\tau + \tau Vd\sigma\tau \\ &= \partial_t\tau\,(xdt - duS\partial_t\tau\partial_u\sigma) + \partial_u\tau\,(xdu - dtS\partial_u\tau\partial_t\sigma). \end{aligned}$$

Let us write

$$S\partial_t\tau\partial_u\sigma \equiv B \text{ and } S\partial_u\tau\partial_t\sigma \equiv C.$$

Then

$$\begin{aligned} f &= \frac{1}{2}\int(xdt - Bdu)(x\delta du - C\delta dt) - (xdu - Cdt)(x\delta dt - B\delta du) \\ &= \frac{1}{2}\int(x^2 - BC)(dt\delta du - du\delta dt), \end{aligned} \tag{3}$$

$$\therefore\ f = (x^2 - BC)\,\phi,$$

or $$\frac{\phi}{f} = \frac{1}{x^2 - BC},$$

but $$\rho_1\rho_2 = -BC,$$

we have $$\frac{\phi}{f} = \frac{1}{(\rho_1 - x)(\rho_2 - x)} \text{ since } \rho_1 + \rho_2 = 0. \tag{4}$$

Therefore we get the following theorem: The measure of density at any point of a ray is equal to the reciprocal of the product of the distances of this point from the foci of the ray. The measure of density of a ray at any point is always real even if the foci are imaginary. For points lying outside of the focal surfaces the measure of density is always positive, while for points inside it is always negative. We can easily see from the expression that it has its least negative value for $x = 0$ or at the middle point, since $\rho_1\rho_2$ is negative, and at the foci it is infinitely great. In the system of rays with imaginary foci the measure of density is always positive, since $\rho_1\rho_2$ is positive, and is a maximum for $x = 0$, *i. e.* in the middle point. If we now take a section of the bundle of rays at another point x', we have for the measure of density at this point

$$\vartheta' = \frac{\phi}{f'},$$

$$\therefore \frac{f}{f'} = \frac{\vartheta'}{\vartheta}.$$

We thus have the theorem that the area of a section of a bundle of rays at any point (the plane of the section being perpendicular to τ) varies inversely as the measure of density. All the points on the different rays of the system which have the same value of the measure of density, lie on a surface which we will call a surface of equal measure of density. Since the measure of density has all possible values, we will have an infinite number of such surfaces. For the equation of a surface with the constant measure of density ϑ, we have

$$x^2 + \rho_1\rho_2 = \frac{1}{\vartheta},$$

$$x = \pm\sqrt{\frac{1}{\vartheta} - \rho_1\rho_2},$$

therefore the equation is

$$\rho = \sigma \pm \tau\sqrt{\frac{1}{\vartheta} - \rho_1\rho_2}.$$

In order that these surfaces be real it is necessary and sufficient that $\frac{1}{\vartheta}$ should lie between $\rho_1\rho_2$ and ∞. For $\vartheta = \infty$ the surface will be real if the foci are real. We see then that the focal surfaces are surfaces of the constant measure of density ∞. The construction of these surfaces is very easy when the focal surfaces are given.

§5.

The Twist of a Ray.

If we are given two lines in space, and as we pass along the second drop perpendiculars from each point to the first, then we will call the angle passed through by the perpendicular from a point a to a point b in the first ray the twist of the second line about the first from a to b.

For consecutive rays the twist for the length of the whole line is two right angles. If the two lines lie in a plane, the twist of the second about the first is zero or two right angles according as the segment of the first does not or does contain the point of intersection.

We have for the perpendicular from $d\sigma + x(\tau + d\tau)$ to τ by §4, eq. (2),

$$d\rho = xd\tau + \tau V d\sigma\tau;$$

at the initial point $x = 0$,

$$d\rho_0 = \tau V d\sigma\tau.$$

For brevity write $B \equiv S\partial_t\tau\partial_u\sigma$, $C \equiv S\partial_u\tau\partial_t\sigma$. Call α_0 the angle made by $d\rho_0$ with the principal line $d\tau_1 = (\partial_t\tau + \partial_u\tau)\,dt$,

$$\cos 2\alpha_0 = \cos^2\alpha_0 - \sin^2\alpha_0 = \frac{S^2 d\tau_1\tau V d\sigma\tau}{(d\tau_1)^2(\tau V\tau d\sigma)^2} - \frac{S^2 d\tau_2\tau V d\sigma\tau}{(d\tau_2)^2(\tau V d\sigma\tau)^2},$$

where $d\tau_2$ is the other principal line $(\partial_t\tau - \partial_u\tau)\,dt_2$,

$$d\rho = \partial_t\tau\,(xdt - Bdu) + \partial_u\tau\,(xdu - Cdt),$$

$$d\rho_0 = -Bdu\partial_t\tau - Cdt\partial_u\tau,$$

$$\frac{S^2 d\tau_1\tau V d\sigma\tau}{(d\tau_1)^2(\tau V d\sigma\tau)^2} = \frac{(Cdt + Bdu)^2}{2\,(B^2du^2 + C^2dt^2)},$$

$$\frac{S^2 d\tau_2\tau V d\sigma\tau}{(d\tau_2)^2(\tau V d\sigma\tau)^2} = \frac{(Cdt - Bdu)^2}{2\,(B^2du^2 + C^2dt^2)},$$

$$\cos 2\alpha_0 = \frac{2BCdudt}{B^2du^2 + C^2dt^2}, \tag{1}$$

$$\sin 2\alpha_0 = \frac{C^2dt^2 - B^2du^2}{B^2du^2 + C^2dt^2}. \tag{2}$$

Let us call β the twist of the second ray about the first when we advance from the initial point a distance x along the first ray, then

$$\tan\beta = -\frac{IVd\rho d\rho_0}{Sd\rho d\rho_0} = \frac{x\,(Cdt^2 - Bdu^2)}{B^2du^2 + C^2dt^2 - x(B + C)\,dudt},$$

$$x = \frac{(B^2du^2 + C^2dt^2)\sin\beta}{(Cdt^2 - Bdu^2)\cos\beta + (B + C)\,dudt\sin\beta}. \tag{3}$$

Multiply both numerator and denominator by $\dfrac{2BC}{C^2dt^2 + B^2du^2}$ and add to and subtract from the denominator $\frac{C^3dt^2 - B^3du^2}{B^2du^2 + C^2dt^2}\cos\beta$. We get

$$x = \frac{2BC\sin\beta}{\frac{(B-C)(B^2du^2+C^2dt^2)\cos\beta + (B+C)\{(C^2dt^2 - B^2du^2)\cos\beta + 2BCdudt\sin\beta\}}{B^2du^2+C^2dt^2}}$$

$$\therefore x = \frac{2BC\sin\beta}{(B-C)\cos\beta + (B+C)\sin(2\alpha_0+\beta)}. \tag{4}$$

Let us regard β as a constant angle and inquire along what rays we get a maximum or a minimum value of x. The maximum and minimum values are evidently given by

$$\sin(2\alpha_0+\beta) = 1 \text{ and } \sin(2\alpha_0+\beta) = -1.$$

Let us put

$$x_1 = \frac{2BC\sin\beta}{(B-C)\cos\beta - (B+C)},\quad x_2 = \frac{2\beta C\sin\beta}{(B-C)\cos\beta + (B+C)}.$$

Calling α the angle made by $d\rho$ with $\partial_t\tau + \partial_u\tau$, we have

$$\beta = \alpha - \alpha_0,$$

$$\frac{1}{x} - \frac{1}{x_1} = \frac{2\cos^2\left(\alpha_0 + \frac{1}{2}\beta - \frac{\pi}{4}\right)(B+C),}{2BC\sin\beta} \tag{5}$$

$$\frac{1}{x_2} - \frac{1}{x} = \frac{2\sin^2\left(\alpha_0 + \frac{\beta}{2} - \frac{\pi}{4}\right)(B+C),}{2BC\sin\beta}, \tag{6}$$

$$\frac{1}{x} = \frac{\cos^2\left(\alpha_0 + \frac{\beta}{2} - \frac{\pi}{4}\right)}{x_2} + \frac{\sin^2\left(\alpha_0 + \frac{\beta}{2} - \frac{\pi}{4}\right)}{x_1}, \tag{7}$$

Thus if we proceed along any ray until its twist about the primitive ray is equal to a constant angle β, this equation gives us the relation between the distance along the primitive ray for this ray and the angle made by this ray with the principal line $d\tau$ and the maximum and minimum distances. This relation becomes, when the rays are normal to a surface, the Eulerian theorem,

$$\frac{1}{R} = \frac{\cos^2\alpha_0}{R_2} + \frac{\sin^2\alpha_0}{R_1},$$

where R_1 and R_2 are the radii of curvature in the principal section, and R the radius of curvature in any normal plane, and α_0 the angle made by this plane with one of the principal planes. In the case of the two focal rays the twist is either zero or two right angles. In the case where the foci are imaginary, the twist cannot be zero for a finite segment of the line, and therefore the twist of the ray cannot change its sense. Thus we have two kinds of systems of rays with imaginary foci, those in which the twist is to the right and those in which it is to the left. When the foci are real, the twist of any ray from one focus to the other is found as follows:

$$d\rho = \partial_t\tau\,(xdt - Bdu) + \partial_u\tau\,(xdu - Cdt)$$

for the foci $x = \pm\sqrt{BC} \equiv \pm A$,

then
$$d\rho_1 = \partial_t\tau\,(Adt - Bdu) + \partial_u\tau\,(Adu - Cdt), \tag{8}$$
$$-d\rho_2 = \partial_t\tau\,(Adt + Bdu) + \partial_u\tau\,(Adu + Cdt). \tag{9}$$

Calling β the angle between $d\rho_1$ and $d\rho_2$, we have

$$\tan^2\beta = -\frac{V^2 d\rho_1 d\rho_2}{S^2 d\rho_1 d\rho_2},$$
$$V^2 d\rho_1 d\rho_2 = 4A^2\,(Bdu^2 - Cdt^2)^2,$$
$$S^2 d\rho_1 d\rho_2 = (B - C)^2(Bdu^2 - Cdt^2)^2,$$
$$\therefore\ \tan^2\beta = \frac{4A^2}{(B-C)^2},$$

and is therefore independent of $\frac{dt}{du}$.

We have the focal lines

$$d\tau_1 \| \sqrt{B}\,\partial_t\tau - \sqrt{C}\,\partial_u\tau,$$
$$d\tau_2 \| \sqrt{B}\,\partial_t\tau + \sqrt{C}\,\partial_u\tau.$$

Calling β_0 the angle between them, we have

$$\tan^2\beta_0 = -\frac{V^2 d\tau_1 d\tau_2}{S^2 d\tau_1 d\tau_2} = \frac{4BC}{(B-C)^2}.$$

Since $A^2 = BC$, we have

$$\beta = \beta_0. \tag{10}$$

Therefore we have the following theorem: The twist of any consecutive ray from focus to focus of the primitive ray is constant and equal to the angle between the focal lines.

§6.

The Sections of the Bundle of Rays.

If we construct at any point of a ray a plane perpendicular to it, we get an infinitesimal curve lying in this plane which is the section of the bundle of consecutive rays. The vector equation of this curve is

$$\rho = xd\tau + \tau V d\sigma\tau,$$

or
$$\rho = \partial_t\tau\,(xdt - Bdu) + \partial_u\tau\,(xdu - Cdt),$$

the origin being the point at which the perpendicular plane is constructed. To see the character of the sections at the foci, we put $x = \pm A$. Writing $x = A$ we have

$$\rho = \partial_t \tau\,(A dt - B du) + \partial_u \tau\,(A du - C dt);$$

but $\dfrac{A}{B} = \dfrac{C}{A}$ since $A^2 = BC$,

$$\therefore\ \rho = \left(\frac{A}{B}\,dt - du\right)(B\partial_t\tau - A\partial_u\tau)$$

$$= \sqrt{B}\left(\frac{A}{B}\,dt - du\right)(\sqrt{B}\partial_t\tau - \sqrt{C}\partial_u\tau).$$

But the focal line $d\tau_1 \| \sqrt{B}\partial_t\tau - \sqrt{C}\partial_u\tau$. Therefore we have the following theorem: The sections at the foci are lines coinciding in direction with the focal lines. The lengths of these lines are zero when

$$\sqrt{B}\partial_t\tau - \sqrt{C}\partial_u\tau = 0,$$

and

$$\sqrt{B}\partial_t\tau + \sqrt{C}\partial_u\tau = 0,$$

$$\therefore\ B = 0,\ C = 0.$$

But since

$$r_1 r_2 = -\left(\frac{S\partial_t\tau\partial_u\sigma + S\partial_u\tau\partial_t\sigma}{2}\right)^2$$

and

$$\rho_1\rho_2 = -BC,$$

we have

$$r_1 = r_2 = \rho_1 = \rho_2 = 0,$$

and therefore in this case the two foci and the two vertices coincide with the centre. Thus we have every consecutive ray going through the centre of the primitive ray. These rays we will call principal rays. Generally there are no principal rays since we have three equations to be satisfied, namely, all the coefficients of §1, eq. (1) must be zero. These conditions are in general only equivalent to two; but there is also another condition, namely, that the foci must be real or

$$S\partial_t\tau\partial_u\sigma = S\partial_u\tau\partial_t\sigma.$$

In case only the two foci coincide with the centre, the section of the bundle of rays is only a straight line at the centre. The section in this case would lie in the plane in which the two focal planes have united. Because the condition that the two foci coincide with the centre only involves one equation between the parameters, there is generally a continuous succession of such lines which form a ruled surface. All rays which are tangent to the curve of intersection of the two focal surfaces have their foci coinciding.

Let us now investigate the character of these sections. Suppose dt and du are connected by the equation

$$B^2du^2 + C^2dt^2 = \varkappa^2,$$

where $\varkappa$ is an infinitesimal constant. We have

$$d\rho = (xdt - Bdu)\,\partial_t\tau + (xdu - Cdt)\,\partial_u\tau .$$

Since $\partial_t\tau$ and $\partial_u\tau$ are unit vectors perpendicular to each other, we may write

$$xdt - Bdu \equiv X,$$

and

$$xdu - Cdt \equiv Y,$$

then

$$dt = \frac{xX + BY}{x^2 - BC} \text{ and } du = \frac{xY + CX}{x^2 - BC}.$$

Therefore the Cartesian equation of any section is

$$C^2\left(\frac{xX + BY}{x^2 - BC}\right)^2 + B^2\left(\frac{xY + CX}{x^2 - BC}\right)^2 = \varkappa^2.$$

Thus we see that under these conditions the section of the bundle of rays is always an ellipse with its centre in τ.

For $x = 0$ this is a circle. Therefore the perpendicular section at the centre of the ray is a circle of radius $\varkappa$.

If we call ϑ the angle made by the major axis on any section with X, we have

$$\tan 2\vartheta = \frac{2BC}{x(C - B)},$$

or, since $\rho_1\rho_2 = -BC$,

$$\tan 2\vartheta = \frac{2\rho_1\rho_2}{x(B - C)};$$

for $x = \rho_1$,

$$\tan 2\vartheta = \frac{2\rho_2}{(B - C)},$$

for $x = \rho_2$,

$$\tan 2\vartheta = \frac{2\rho_1}{(B - C)}.$$

These give us the tangent of double the angle made by the focal lines with $\partial_t\tau$. As we shall see further on, in the case when the rays are normal to a surface, $B = C$,

$$\therefore \tan 2\vartheta = \infty,$$

$$\vartheta = 45^\circ,$$

$\therefore$ the principal lines are the principal axes of all sections in this case. Let us now obtain the expressions for the lengths of the axes in any case. When the equation of any ellipse is in the form

$$a_{11}x^2 + a_{22}y^2 + a_{33} + 2a_{12}xy + 2a_{13}x + 2a_{23}y = 0,$$

the squares of the semi-axes are given by $-\dfrac{A}{\lambda_1 A_{33}}$ and $\dfrac{A}{-\lambda_2 A_{33}}$, where

$$A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \quad A_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix}.$$

$a_{13} = a_{31}$, etc., and λ_1 and λ_2 are the roots of

$$\lambda^2 - (a_{11} + a_{12})\,\lambda + a_{11}a_{22} - a_{12}^2 = 0.$$

In our case we get

$$\lambda^2 - \{(B^2 + C^2)\,x^2 + 2B^2C^2\}\,\lambda + x^2(x^2 - BC)^2 = 0,$$

$$\frac{A}{A_{33}} = -x^2(x^2 - BC)^2,$$

we have
$$\lambda = \frac{(B^2 + C^2)\,x^2 + 2B^2C^2 \pm x(B + C)\sqrt{x^2(B - C)^2 + 4B^2C^2}}{2}.$$

If we write

$$2\lambda_1 \equiv (B^2 + C^2)\,x^2 + 2B^2C^2 - x(B + C)\sqrt{x^2(B - C)^2 + 4B^2C^2},$$

and
$$2\lambda_2 \equiv (B^2 + C^2)\,x^2 + 2B^2C^2 + x(B + C)\sqrt{x^2(B - C)^2 + 4B^2C^2},$$

for the first focus $x = \sqrt{BC}$.

Substituting this value of x in the expressions for the two axes, we have

$$-\frac{A}{\lambda_2 A_{33}} = 0.$$

We have
$$-\frac{A}{\lambda_1 A_{33}} = \frac{2x^2(x^2 - BC)^2}{(B^2 + C^2)\,x^2 + 2B^2C^2 - x(B + C)\sqrt{x^2(B - C)^2 + 4B^2C^2}},$$

which is indeterminate, being of form $\dfrac{0}{0}$.

Differentiating numerator and denominator twice, and substituting $x = \sqrt{BC}$, we obtain
$$-\frac{A}{\lambda_1 A_{33}} = \frac{x^2(B + C)^2}{BC},$$

or the length of the major axis is $\dfrac{2x(B + C)}{\sqrt{BC}}$, the length of the minor axis being zero. The lengths of the sections at the foci are therefore in this case, viz. when the section at the centre is a circle of radius $\varkappa$,

$$\frac{2\varkappa(B+C)}{\rho_1} \text{ and } \frac{2\varkappa(B+C)}{\rho_2},$$

we have

$$d\rho = \partial_t\tau\,(xdt - Bdu) + \partial_u\tau\,(xdu - Cdt);$$

at the foci $x = \pm\sqrt{BC}$,

$$\therefore\ d\tau_1 = (\sqrt{C}dt - \sqrt{B}du)\{\sqrt{B}\partial_t\tau - \sqrt{C}\,\partial_u\tau\},$$

$$d\tau_2 = (\sqrt{C}dt + \sqrt{B}du)\{\sqrt{B}\partial_t\tau + \sqrt{C}\partial_u\tau\}.$$

These are the focal lines, and it is interesting to observe that through each point on the focal line two rays pass, namely, the two rays corresponding to the values of dt and du which we get from the two equations

$$\sqrt{C}dt - \sqrt{B}du = \text{constant},$$

or

$$\sqrt{C}dt + \sqrt{B}du = \text{"}$$

and

$$B^2du^2 + C^2dt^2 = \varkappa^2;$$

for a ray passing through the first focus

$$\sqrt{C}dt - \sqrt{B}du = 0,$$

or

$$\frac{dt}{du} = \frac{\sqrt{B}}{\sqrt{C}}.$$

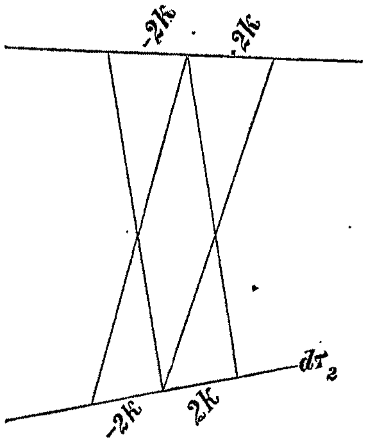
Substituting in $B^2du^2 + C^2dt^2 = \varkappa^2$, we get the two pairs of values

$$dt = \frac{\varkappa}{\sqrt{C(B+C)}},\quad du = \frac{\varkappa}{\sqrt{B(B+C)}}$$

and

$$dt = -\frac{\varkappa}{\sqrt{C(B+C)}},\quad du = -\frac{\varkappa}{\sqrt{B(B+C)}}.$$

We may inquire where these rays cut the second focal line $d\tau_2$. Substituting in $d\tau_2$ we get the lengths to lie $2\varkappa$ and $-2\varkappa$. The case is then thus:



To obtain the values of dt and du which give us the rays passing through the extremities of the focal lines, we have for $d\tau_1$

$$\sqrt{C}\delta dt - \sqrt{B}\delta du = 0$$

and $B^2 du\delta du + C^2 dt\delta dt = 0$,

$$\therefore \frac{dt}{du} = -\frac{B\sqrt{B}}{C\sqrt{C}}.$$

Substituting in $B^2du^2 + C^2dt^2 = \varkappa^2$, we get

$$dt = -\frac{\sqrt{B}\varkappa}{C\sqrt{B+C}},\quad du = \frac{\varkappa\sqrt{C}}{B\sqrt{B+C}}$$

and

$$dt = +\frac{\sqrt{B}\varkappa}{C\sqrt{B+C}},\quad du = -\frac{\varkappa\sqrt{C}}{B\sqrt{B+C}}.$$

For the other focal line

$$\sqrt{C}\delta dt + \sqrt{B}\delta du = 0$$

and

$$B^2 du\delta du + C^2 dt\delta dt = 0,$$

$$\therefore \frac{dt}{du} = \frac{B\sqrt{B}}{C\sqrt{C}},$$

and substituting in $B^2du^2 + C^2dt^2 = \varkappa^2$, we get

$$dt = \frac{\varkappa\sqrt{B}}{C\sqrt{B+C}},\quad du = \frac{\varkappa\sqrt{C}}{B\sqrt{B+C}}$$

and

$$dt = -\frac{\varkappa\sqrt{B}}{C\sqrt{B+C}},\quad du = -\frac{\varkappa\sqrt{C}}{B\sqrt{B+C}}.$$

We thus see that the rays passing through the extremities of the focal lines do not coincide. Let us now find where the ray

$$dt = \frac{\varkappa\sqrt{B}}{C\sqrt{B+C}},\quad du = -\frac{\varkappa\sqrt{C}}{B\sqrt{B+C}},$$

passing through the extremity of $d\tau_1$, cuts $d\tau_2$. We have for the distance of this point from the second focus

$$\sqrt{B+C}\,(\sqrt{C}dt + \sqrt{B}du).$$

In this case we have this distance

$$\sqrt{B+C}\left\{\frac{\varkappa\sqrt{B}}{\sqrt{C(B+C)}} - \frac{\sqrt{C}\varkappa}{\sqrt{B(B+C)}}\right\} = \varkappa\frac{B-C}{\sqrt{BC}}.$$

course a whole family of surfaces
equation of condition between t and u. When $S\partial_t\tau\partial_u\sigma = S\partial_u\tau\partial_t\sigma$, the roots of equation (1), §1 are the same as those of (2), §2, and also those of (2), §1 the same as those of (3), §2. Therefore, in those systems whose rays are normals to a surface, the focal and principal planes coincide, as do also the foci and vertices. If in this case we choose any surface which cuts the rays normally, we have the distances of the foci from the initial point equal to the two principal radii of curvature, and also have the theorem

$$r = r_1 \cos^2\omega + r_2 \sin^2\omega.$$

If we take the bundle of normals consecutive to the given normal, then we have seen that the perpendicular sections at the foci are in the direction of the focal lines. Therefore all the normals consecutive to a given normal pass through two lines, one perpendicular to the normal at one principal centre of curvature and the other at the other, and also since in this case the focal lines coincide with the principal lines, these two lines lie in the planes of principal section. In the case we are considering the principal and focal surfaces coincide and become the surfaces treated by Monge, namely, the loci of the principal centres of curvature. For the lines of curvature of the normal surface we have

$$S\tau d\tau d\rho = 0,$$

since τ is normal to the surface and this is the condition that a consecutive normal cut the primitive one, but

$$d\rho = d\sigma + x_1 d\tau + dx_1\tau,$$

$\therefore$ condition is $S\tau d\sigma d\tau = 0$, and $\therefore$ we have the theorem that the focal lines give the directions of the lines of curvature on the surface. The umbilics of a surface, or the points at which the normal is cut by all consecutive normal and consequently the two principal centres of curvature coincide, correspond to the initial point on a principal ray in the general theory. Also in

[redacted]

$$dt = -\frac{}{\sqrt{B^2 + C}}$$

$$dt = \frac{\varkappa}{\sqrt{B^2 + C^2}}, \quad du = \frac{}{\sqrt{B^2 + C^2}},$$

$$dt = -\frac{\varkappa}{\sqrt{B^2 + C^2}}, \quad du = \frac{\varkappa}{\sqrt{B^2 + C^2}}. \qquad \text{IV.}$$

For the points where I cuts the focal lines we have

$$\sqrt{B + C}\{\sqrt{C}dt - \sqrt{B}du\}$$

and

$$\sqrt{B + C}\{\sqrt{C}dt + \sqrt{B}du\},$$

and substituting, we have

$$\frac{\varkappa\sqrt{B + C}}{\sqrt{B^2 + C^2}}(\sqrt{C} - \sqrt{B}) \text{ and } \frac{\varkappa\sqrt{B + C}}{\sqrt{B^2 + C^2}}\{\sqrt{C} + \sqrt{B}\}.$$

For the ratio of the lengths cut off on the focal lines by any ray we have, calling δ that on one and ε that on other,

$$\frac{\delta}{\varepsilon} = \frac{\sqrt{C}dt - \sqrt{B}du}{\sqrt{C}dt + \sqrt{B}du}.$$

§7.

Systems of Rays Normal to a Surface.

Let $\rho = \sigma + x\tau$, where x is a function of t and u, be the equation of any surface to which the rays are normal. Then we must have

$$S\tau\partial_t\rho = 0 \text{ and } S\tau\partial_u\rho = 0,$$

$$\partial_t\rho = \partial_t\sigma + x\partial_t\tau + \frac{\partial x}{\partial t}\tau,$$

$$\partial_u\rho = \partial_u\sigma + x\partial_u\tau + \frac{\partial x}{\partial u}\tau,$$

$$S\tau\partial_t\rho = S\tau\partial_t\sigma - \frac{\partial x}{\partial t} = 0, \text{ since } \tau^2 = -1 \text{ and } S\tau\partial_t\tau = 0,$$

$$S\tau\partial_u\rho = S\tau\partial_u\sigma - \frac{\partial x}{\partial u} = 0,$$

the case considered the measure of density coincides exactly with Gauss' measure of absolute curvature. Our formula for the measure of density becomes the reciprocal of the product of the distance of the point from the foci; in the case of normals, the measure of density at the initial point is the reciprocal of the product of the principal radii of curvature.

In conclusion I would state that the arrangement of this paper is like that of Kummer, and also that I am indebted to Dr. Story of the Johns Hopkins University for many valuable suggestions.

Some Theorems concerning the Centre of Gravity.

By F. Franklin.

Lagrange's two theorems on the centre of gravity may be expressed by the equations

$$M\Sigma m_a\,(GA)^2 = \Sigma m_a m_b\,(AB)^2, \tag{1}$$

$$\Sigma m_a\,(OA)^2 = \Sigma m_a\,(GA)^2 + M(OG)^2, \tag{2}$$

where G is the centre of gravity of the particles of mass m_a, m_b, ... situated at the points A, B, ... and M is the sum of the masses. These theorems admit of extensions, in which, instead of the distances GA, AB, ..., there appear the areas of the triangles GAB, ABC, ...; or the volumes of the tetrahedra $GABC$, $ABCD$, ...; and so on, in space of more than three dimensions. Namely, we have the series of relations

$$\left.\begin{array}{r} M\Sigma m_a\,(GA)^2 = \Sigma m_a m_b\,(AB)^2, \\ M\Sigma m_a m_b\,(GAB)^2 = \Sigma m_a m_b m_c\,(ABC)^2, \\ M\Sigma m_a m_b m_c\,(GABC)^2 = \Sigma m_a m_b m_c m_d\,(ABCD)^2, \\ \text{etc.} \qquad\qquad\qquad\qquad\qquad \end{array}\right\} \quad \text{I.}$$

$$\left.\begin{array}{r} \Sigma m_a\,(OA)^2 = \Sigma m_a\,(GA)^2 + M(OG)^2, \\ \Sigma m_a m_b\,(OAB)^2 = \Sigma m_a m_b\,(GAB)^2 + M\Sigma m_a\,(OGA)^2, \\ \Sigma m_a m_b m_c\,(OABC)^2 = \Sigma m_a m_b m_c\,(GABC)^2 + M\Sigma m_a m_b\,(OGAB)^2, \\ \text{etc.} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \end{array}\right\} \quad \text{II.}$$

These theorems may be proved almost instantaneously. I shall give the proof for the case of triangles, any other case being obviously demonstrable in the same way.

I. Let the origin of coordinates be the centre of gravity, and consider the matrices

$$\left|\begin{array}{lllll} m_a & m_b & m_c & m_d & \ldots . \\ m_a y_a & m_b y_b & m_c y_c & m_d y_d & \ldots . \\ m_a z_a & m_b z_b & m_c z_c & m_d z_d & \ldots . \end{array}\right|, \quad \left|\begin{array}{lllll} 1 & 1 & 1 & 1 & \ldots . \\ y_a & y_b & y_c & y_d & \ldots . \\ z_a & z_b & z_c & z_d & \ldots . \end{array}\right|,$$

and the similar pairs of matrices in z, x and x, y. The sum of the products of corresponding determinants in these matrices is evidently $4\Sigma m_a m_b m_c (ABC)^2$. On the other hand, multiplying the pair of matrices above written, we get

$$\begin{vmatrix} M, & 0 & , & 0 & , \\ 0\ , & \Sigma m_a y_a^2 & , & \Sigma m_a y_a z_a, \\ 0, & \Sigma m_a y_a z_a, & \Sigma m_a z_a^2 & , \end{vmatrix} = M \begin{vmatrix} m_a y_a, & m_b y_b, & m_c y_c, & \dots \\ m_a z_a\ , & m_b z_b\ , & m_c z_c\ , & \dots \end{vmatrix} \cdot \begin{vmatrix} y_a, & y_b, & y_c, & \dots \\ z_a\ , & z_b\ , & z_c\ , & \dots \end{vmatrix}.$$

Now, since G is the origin, the determinant $\begin{vmatrix} y_a & y_b \\ z_a & z_b \end{vmatrix}$ is twice the projection, on the plane of yz, of the triangle GAB; hence the sum of the products of corresponding determinants in the last-written pair of matrices and in the similar pairs in z, x and x, y is $4M\Sigma m_a m_b (GAB)^2$. Therefore

$$M\Sigma m_a m_b (GAB)^2 = \Sigma m_a m_b m_c (ABC)^2.$$

II. Let the origin be at any point O; let the coordinates of G be x, y, z; let those of A be $x + x_a$, $y + y_a$, $z + z_a$, etc. Then on the one hand

$$\begin{vmatrix} m_a(y+y_a), & m_b(y+y_b), & m_c(y+y_c), & \dots \\ m_a(z+z_a)\ , & m_b(z+z_b), & m_c(z+z_c), & \dots \end{vmatrix} \cdot \begin{vmatrix} y+y_a, & y+y_b, & y+y_c, & \dots \\ z+z_a, & z+z_b, & z+z_c, & \dots \end{vmatrix},$$

added to the two similar products, gives $4\Sigma m_a m_b (OAB)^2$. On the other hand, if we apply the process for the multiplication of matrices, and bear in mind that $\Sigma m_a y_a = 0$, $\Sigma m_a z_a = 0$, it is plain that the product of the above two matrices is equivalent to that of the two following:

$$\begin{vmatrix} m_a y, & m_a y_a, & m_b y, & m_b y_b, & m_c y, & m_c y_c, & \dots \\ m_a z\ , & m_a z_a\ , & m_b z\ , & m_b z_b\ , & m_c z\ , & m_c z_c\ , & \dots \end{vmatrix}, \begin{vmatrix} y, & y_a, & y, & y_b, & y, & y_c, & \dots \\ z\ , & z_a\ , & z\ , & z_b\ , & z\ , & z_c\ , & \dots \end{vmatrix}.$$

And the product of these, added to the two similar products, evidently gives $4\{\Sigma m_a m_b (GAB)^2 + M\Sigma m_a (OGA)^2\}$. Hence

$$\Sigma m_a m_b (OAB)^2 = \Sigma m_a m_b (GAB)^2 + M\Sigma m_a (OGA)^2.$$

Any theorem under I is a sufficiently obvious corollary of the corresponding theorem under II; but it seemed worth while to give an independent proof of the former. The method of that proof gives rise also to a theorem concerning the principal moments of a system at the centre of gravity which seems not without interest. If, besides taking the origin at the centre of gravity, we take for axes of coordinates, the principal axes of the system of particles, the multiplication of the matrices

$$\left|\begin{array}{lllll} m_a\ , & m_b\ , & m_c\ , & m_d\ , & \ldots \\ m_a x_a, & m_b x_b, & m_c x_c, & m_d x_d, & \ldots \\ m_a y_a, & m_b y_b, & m_c y_c, & m_d y_d, & \ldots \\ m_a z_a\ , & m_b z_b\ , & m_c z_c\ , & m_d z_d\ , & \ldots \end{array}\right| , \quad \left|\begin{array}{lllll} 1\ , & 1\ , & 1\ , & 1\ , & \ldots \\ x_a, & x_b, & x_c, & x_d, & \ldots \\ y_a, & y_b, & y_c, & y_d, & \ldots \\ z_a\ , & z_b\ , & z_c\ , & z_d\ , & \ldots \end{array}\right| ,$$

gives the equation

$$M\Sigma m_a x_a^2 . \Sigma m_a y_a^2 . \Sigma m_a z_a^2 = 36\Sigma m_a m_b m_c m_d (ABCD)^2.$$

Treating in the same way the similar pairs of matrices with three and two rows respectively, and writing

$$\Sigma m_a x_a^2 = A', \ \Sigma m_a y_a^2 = B', \ \Sigma m_a z_a^2 = C',$$

$$\Sigma m_a m_b (AB)^2 = \Sigma_2, \ 2^2\Sigma m_a m_b m_c (ABC)^2 = \Sigma_3, \ 6^2\Sigma m_a m_b m_c m_d (ABCD)^2 = \Sigma_4,$$

we have

$$M(A' + B' + C') = \Sigma_2, \ M(B'C' + C'A' + A'B') = \Sigma_3, \ MA'B'C' = \Sigma_4,$$

so that A', B', C' are the roots of the equation

$$M\lambda^3 - \Sigma_2\lambda^2 + \Sigma_3\lambda - \Sigma_4 = 0.$$

Thus the principal moments at the centre of gravity are determined in terms of the masses and mutual distances of the particles. Of course a similar equation holds for any number of dimensions.

The moments of inertia themselves are $B' + C'$, $C' + A'$, $A' + B'$, and the equation of which these are the roots is

$$M^2\mu^3 - 2M\Sigma_2\mu^2 + (\Sigma_2^2 + M\Sigma_3)\mu - (\Sigma_2\Sigma_3 - M\Sigma_4) = 0.$$